M1 General Physics **2023-2024**

Particles

Exam

Second session

February 15th 2024

Documents allowed

Notes:

- **The subject is deliberately long**. Solving at least one of the two problems will ensure a good mark!
- One may use the usual system of units in which $c = 1$ and $\hbar = 1$.
- Space coordinates may be freely denoted as (x, y, z) or (x^1, x^2, x^3).
- Any drawing, at any stage, is welcome, and will be rewarded!

1 Green function and covariant gauge

1. Green function (scalar case)

Suppose we want to solve the equation

$$\Box\psi(x) = \phi(x)\,, \tag{1}$$

where $\phi(x)$ is an arbitrary known function, named source term, and $\Psi(x)$ is the unknown quantity we are looking for.

A very efficient way to solve this problem is to determine the Green's function $G(x)$ solution of the auxiliary problem

$$\Box G(x) = \delta^{(4)}(x)\,. \tag{2}$$

At this stage, the fact of being in a Minkowski space plays no role.

Show that the solution to the problem (1) is then formally obtained as a convolution of the source with Green's function:

$$\psi(x) = \int d^4x'\, G(x - x')\, \phi(x')\,. \tag{3}$$

We will now extend the previous discussion to the case of a field of spin 1.

2. Transverse and longitudinal projectors.

For any momentum p (assumed to be non light-like), we introduce the operators L and T defined through their matrix elements as

$$L^\mu{}_\nu = \frac{p^\mu p_\nu}{p^2}\,, \tag{4}$$

$$T^\mu{}_\nu = g^\mu{}_\nu - \frac{p^\mu p_\nu}{p^2}\,. \tag{5}$$

What are their algebraic properties? One should study in detail the operators L^2, T^2, LT, TL and $L + T$: compute their matrix elements and conclude about their nature.
Compute the action of the operators L and T on p. Deduce the kernels of these two operators. Finally, characterize more precisely the two operators L and T.

In the case of electromagnetism, Maxwell's equation satisfied by the vector potential is written as

$$\Box A^\mu - \partial^\mu(\partial.A) = j^\mu \tag{6}$$

3. Show that it is not sufficient on its own to determine A^μ as a function of j^μ. For that, one should consider the Fourier conjugated equation, and show that one is forced to invert the operator T. Conclude.

To solve this problem, we have to give the photon a mass, or add a gauge-fixing term to the Lagrangian. In the covariant Lorentz gauge, we add the term

$$\mathcal{L}_{jauge} = \frac{\lambda}{2}(\partial.A)^2\,, \tag{7}$$

so that the full Lagrangian can be written as

$$\mathcal{L}_{e.m} = -\frac{1}{4}F^2 - j.A + \frac{\lambda}{2}(\partial.A)^2\,, \tag{8}$$

4. Write the equations of motion of the corresponding Lagrangian.

5. We are looking for the Green's function $G_{F\mu\nu}$. In comparison to Eq. (2), since the field A^μ carries a Lorentz index, the right-hand-side should now contains, in addition to the Dirac distribution, the identity in Minkowski space.

We denote

$$M_{\mu\nu} = p^2 g_{\mu\nu} - (1+\lambda)p_\mu p_\nu\,. \tag{9}$$

(i) Explain why, in Fourier space, $\tilde{G}_{F\mu\nu}$ is solution of the equation

$$M_{\mu\nu}\tilde{G}_F{}^\nu{}_\rho = -g_{\mu\rho}\,, \tag{10}$$

(ii) We want now to solve Eq. (10). Using the two operators L and T, and calculating the inverse operator of M, show that Green's function, corresponding to the equation of motion obtained in question 4 can be written in the form (using the so-called Feynman's prescription $p^2 \to p^2 + i\epsilon$ with $\epsilon \to 0^+$ to regulate the pole at $p^2 = 0$, which play no role here),

$$G_{F\mu\nu}(x-y,\lambda) = -\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip\cdot(x-y)}\frac{1}{p^2+i\epsilon}\left(g_{\mu\nu} - \frac{1+\lambda}{\lambda}\frac{p_\mu p_\nu}{p^2}\right)\,. \tag{11}$$

6. Discuss the limits $\lambda \to 0$ and $\lambda \to \infty$. In this second case, relate the properties of the propagator obtained to the form of the Lagrangian.

7. What is the contribution of the second term in the parenthesis in Eq. (11), when contracted with a conserved current?

2 Acceleration in special relativity

Consider a frame R' traveling with speed $\beta = v$ ($c = 1$) with respect to the frame R along the x-axis.

We denote by $\vec{u} = (u_x, u_y, u_z)$ the velocity of a particle in frame R and $\vec{u}\,' = (u'_x, u'_y, u'_z)$ the corresponding acceleration of this particle in frame R'.

1. Briefly show that

$$\begin{aligned} u'_x &= \frac{u_x - \beta}{1 - \beta u_x} && (12) \\ u'_y &= \frac{1}{\gamma}\frac{u_y}{1 - \beta u_x} && (13) \\ u'_z &= \frac{1}{\gamma}\frac{u_z}{1 - \beta u_x}\,. && (14) \end{aligned}$$

2. The denote by $\vec{a} = (a_x, a_y, a_z)$ the velocity of a particle in frame R and $\vec{a}\,' = (a'_x, a'_y, a'_z)$ the corresponding velocity of in frame R'.

Show that

$$\begin{aligned} a'_x &= \frac{a_x}{\gamma^3(1 - \beta u_x)^3} && (15) \\ a'_y &= \frac{a_y}{\gamma^2(1 - \beta u_x)^2} + \frac{\beta\, u_y\, a_x}{\gamma^2(1 - \beta u_x)^3} && (16) \\ a'_z &= \frac{a_z}{\gamma^2(1 - \beta u_x)^2} + \frac{\beta\, u_z\, a_x}{\gamma^2(1 - \beta u_x)^3}\,. && (17) \end{aligned}$$

3. Starting from the 4-velocity

$$U^\mu = \frac{dX^\mu}{d\tau} = \gamma(u)(1, \vec{u}) \qquad (18)$$

with $\gamma(u) = 1/\sqrt{1 - u^2}$, we define the 4-vector acceleration as

$$A^\mu = \frac{dU^\mu}{d\tau}\,. \qquad (19)$$

(i) Prove that

$$\frac{d\gamma(u)}{dt} = \gamma(u)^3 \vec{u} \cdot \vec{a}\,. \qquad (20)$$

(ii) Show that

$$A^\mu = (\gamma^4 \vec{u} \cdot \vec{a}, \gamma^2 \vec{a} + \gamma^4 (\vec{u} \cdot \vec{a}) \vec{u}) \,, \tag{21}$$

where $\gamma = \gamma(u)$.

(iii) Show that A^μ is in general a space-like 4-vector.

Hint: work in the rest frame.

4. Orthogonality of U and A.

(i) What is the value of U^2? Deduce that U and A are orthogonal.

(ii) Obtain this result by working in an appropriate frame.

FIN !