M1 General Physics **2023-2024**

Particles

Exam

Second session

February 15th 2024

Documents allowed

Notes:

- **The subject is deliberately long**. Solving at least one of the two problems will ensure a good mark!
- One may use the usual system of units in which $c = 1$ and $\hbar = 1$.
- Space coordinates may be freely denoted as (x, y, z) or (x^1, x^2, x^3).
- Any drawing, at any stage, is welcome, and will be rewarded!

1 Green function and covariant gauge

1. Green function (scalar case)
Suppose we want to solve the equation

$$\Box\psi(x) = \phi(x)\,, \tag{1}$$

where $\phi(x)$ is an arbitrary known function, named source term, and $\Psi(x)$ is the unknown quantity we are looking for.
A very efficient way to solve this problem is to determine the Green's function $G(x)$ solution of the auxiliary problem

$$\Box G(x) = \delta^{(4)}(x)\,. \tag{2}$$

At this stage, the fact of being in a Minkowski space plays no role.
Show that the solution to the problem (1) is then formally obtained as a convolution of the source with Green's function:

$$\psi(x) = \int d^4x'\, G(x - x')\,\phi(x')\,. \tag{3}$$

Solution

Indeed,

$$\Box_x\psi(x) = \int d^4x'\,\Box_x G(x - x')\,\phi(x') = \int d^4x'\,\delta^{(4)}(x - x')\,\phi(x') = \phi(x)\,. \tag{4}$$

We will now extend the previous discussion to the case of a field of spin 1.

2. Transverse and longitudinal projectors.

For any momentum p (assumed to be non light-like), we introduce the operators L and T defined through their matrix elements as

$$L^{\mu}{}_{\nu} = \frac{p^{\mu}p_{\nu}}{p^2}, \tag{5}$$

$$T^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} - \frac{p^{\mu}p_{\nu}}{p^2}. \tag{6}$$

What are their algebraic properties? One should study in detail the operators L^2, T^2, LT, TL and $L+T$: compute their matrix elements and conclude about their nature.
Compute the action of the operators L and T on p. Deduce the kernels of these two operators. Finally, characterize more precisely the two operators L and T.

Solution

These two operators are projectors, since $L^2 = L$ and $T^2 = T$. Indeed,

$$L_{\mu\rho}L^{\rho}{}_{\nu} = \frac{p_{\mu}p_{\rho}}{p^2}\frac{p^{\rho}p_{\nu}}{p^2} = \frac{p_{\mu}p_{\nu}}{p^2} = L_{\mu\nu}$$

and

$$T_{\mu\rho}T^{\rho}{}_{\nu} = \left(g_{\mu\rho} - \frac{p_{\mu}p_{\rho}}{p^2}\right)\left(g^{\rho}_{\nu} - \frac{p^{\rho}p_{\nu}}{p^2}\right) = g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} - \frac{p_{\mu}p_{\nu}}{p^2} + p^2\frac{p_{\mu}p_{\nu}}{p^2p^2} = g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} = T_{\mu\nu}.$$

Their sum is equal to the identity, since

$$L_{\mu\nu} + T_{\mu\nu} = g_{\mu\nu}.$$

Finally, they are orthogonal to each other, indeed

$$T_{\mu\rho}L^{\rho}{}_{\nu} = \left(g_{\mu\rho} - \frac{p_{\mu}p_{\rho}}{p^2}\right)\frac{p^{\rho}p_{\nu}}{p^2} = \frac{p_{\mu}p_{\nu}}{p^2} - p^2\frac{p_{\mu}p_{\nu}}{p^2p^2} = 0,$$

and similarly

$$L_{\mu\rho}T^{\rho}{}_{\nu} = \frac{p_{\mu}p_{\rho}}{p^2}\left(g^{\rho}_{\nu} - \frac{p^{\rho}p_{\nu}}{p^2}\right) = \frac{p_{\mu}p_{\nu}}{p^2} - p^2\frac{p_{\mu}p_{\nu}}{p^2p^2} = 0.$$

Finally, one has

$$L^{\mu}{}_{\nu}p^{\nu} = p^{\mu}$$

and

$$T^{\mu}{}_{\nu}p^{\nu} = 0.$$

The projector T projects onto the hyperplane orthogonal to p, and L projects onto the vector line generated by p. The kernel of T is therefore the vector line generated by p, and the kernel of L is the hyperplane orthogonal to p.

In the case of electromagnetism, Maxwell's equation satisfied by the vector potential is written as

$$\Box A^\mu - \partial^\mu(\partial.A) = j^\mu \tag{7}$$

3. Show that it is not sufficient on its own to determine A^μ as a function of j^μ. For that, one should consider the Fourier conjugated equation, and show that one is forced to invert the operator T. Conclude.

Solution

By Fourier transform we have

$$-p^2 A^\mu + p^\mu p_\nu A^\nu = \tilde{j}^\mu$$

thus

$$-p^2 \left[g^\mu{}_\nu - \frac{p^\mu p_\nu}{p^2} \right] A^\nu = \tilde{j}^\mu .$$

i.e.

$$-p^2 T^\mu{}_\nu A^\nu = \tilde{j}^\mu .$$

Now the kernel of T is the vector line carried by p. It is therefore not invertible, since its kernel is non zero.

To solve this problem, we have to give the photon a mass, or add a gauge-fixing term to the Lagrangian. In the covariant Lorentz gauge, we add the term

$$\mathcal{L}_{jauge} = \frac{\lambda}{2}(\partial.A)^2 , \tag{8}$$

so that the full Lagrangian can be written as

$$\mathcal{L}_{e.m} = -\frac{1}{4}F^2 - j.A + \frac{\lambda}{2}(\partial.A)^2 , \tag{9}$$

4. Write the equations of motion of the corresponding Lagrangian.

Solution

The Euler-Lagrange equation reads

$$\frac{\delta \mathcal{L}_{e.m}}{\delta A_\nu} - \partial_\mu \frac{\delta \mathcal{L}_{e.m}}{\delta \partial_\mu A_\nu} = 0 .$$

One has

$$\frac{\delta \mathcal{L}_{e.m}}{\delta \partial_\mu A_\nu} = -F^{\mu\nu} + \lambda \, (\partial \cdot A) g^{\mu\nu}$$

and

$$\frac{\delta \mathcal{L}_{e.m}}{\delta A_\nu} = -j^\nu$$

so that

$$\partial^2 A^\nu - \partial^\nu(\partial \cdot A) - j^\nu - \lambda \partial^\nu(\partial \cdot A) = 0$$

and thus finally

$$\left[\Box g^\mu_\nu - (1+\lambda)\partial^\mu \partial_\nu\right] A^\nu = j^\nu .$$

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5. We are looking for the Green's function $G_{F\mu\nu}$. In comparison to Eq. (2), since the field A^μ carries a Lorentz index, the right-hand-side should now contains, in addition to the Dirac distribution, the identity in Minkowski space.

We denote

$$M_{\mu\nu} = p^2 g_{\mu\nu} - (1+\lambda) p_\mu p_\nu \,. \tag{10}$$

(i) Explain why, in Fourier space, $\tilde{G}_{F\mu\nu}$ is solution of the equation

$$M_{\mu\nu} \tilde{G}_F{}^\nu{}_\rho = -g_{\mu\rho} \,, \tag{11}$$

Solution

Replacing j^μ by g_{μ_ρ} in the equation of motion obtained in the previous question, the result is obvious after Fourier transforming each term.

(ii) We want now to solve Eq. (11). Using the two operators L and T, and calculating the inverse operator of M, show that Green's function, corresponding to the equation of motion obtained in question 4 can be written in the form (using the so-called Feynman's prescription $p^2 \to p^2 + i\epsilon$ with $\epsilon \to 0^+$ to regulate the pole at $p^2 = 0$, which play no role here),

$$G_{F\mu\nu}(x-y, \lambda) = -\frac{1}{(2\pi)^4} \int d^4p \, e^{-ip\cdot(x-y)} \frac{1}{p^2 + i\epsilon} \left(g_{\mu\nu} - \frac{1+\lambda}{\lambda} \frac{p_\mu p_\nu}{p^2} \right) . \tag{12}$$

Solution

The equation of motion is Fourier-transformed to

$$M_{\mu\nu} \tilde{G}_F{}^\nu{}_\rho = -g_{\mu\rho} \,,$$

with

$$M_{\mu\nu} = p^2 \left[(L+T)_{\mu\nu} - (1+\lambda) L_{\mu\nu} \right] = p^2 \left[-\lambda L_{\mu\nu} + T_{\mu\nu} \right] .$$

Using Feynman's prescription, we have

$$\begin{aligned}(M^{-1})_{\mu\nu} &= \left[-\frac{1}{\lambda} L_{\mu\nu} + T_{\mu\nu} \right] \frac{1}{p^2 + i\epsilon} \\ &= \left[-\frac{1}{\lambda} \frac{p_\mu p_\nu}{p^2} + g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] \frac{1}{p^2 + i\epsilon} \\ &= \left[g_{\mu\nu} - \frac{1+\lambda}{\lambda} \frac{p_\mu p_\nu}{p^2} \right] \frac{1}{p^2 + i\epsilon} ,\end{aligned}$$

so that finally

$$G_{F\mu\nu}(x-y, \lambda) = -\frac{1}{(2\pi)^4} \int d^4p \, e^{-ip\cdot(x-y)} \frac{1}{p^2 + i\epsilon} \left(g_{\mu\nu} - \frac{1+\lambda}{\lambda} \frac{p_\mu p_\nu}{p^2} \right) .$$

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6. Discuss the limits $\lambda \to 0$ and $\lambda \to \infty$. In this second case, relate the properties of the propagator obtained to the form of the Lagrangian.

Solution

Case $\lambda \to 0$: the propagator is not defined, so we are back to the original problem.
Case $\lambda \to \infty$: the propagator becomes

$$G_{F\mu\nu}(x-y, \infty) = -\frac{1}{(2\pi)^4} \int d^4p \, e^{-ip\cdot(x-y)} \frac{1}{p^2 + i\epsilon} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) .$$

In this case, the propagator is automatically orthogonal to p^μ. This is expected, since in the Lagrangian, the gauge-fixing term is then multiplied by an infinite factor, requiring the Lorenz gauge to be satisfied.

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7. What is the contribution of the second term in the parenthesis in Eq. (12), when contracted with a conserved current?

Solution

For a conserved current, $p_\mu \tilde{j}^\mu = 0$, thus the second term does not contribute.

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2 Acceleration in special relativity

Consider a frame R' traveling with speed $\beta = v$ $(c = 1)$ with respect to the frame R along the x-axis.

We denote by $\vec{u} = (u_x, u_y, u_z)$ the velocity of a particle in frame R and $\vec{u}\,' = (u'_x, u'_y, u'_z)$ the corresponding acceleration of this particle in frame R'.

1. Briefly show that

$$\begin{aligned} u'_x &= \frac{u_x - \beta}{1 - \beta u_x} &(13)\\ u'_y &= \frac{1}{\gamma}\frac{u_y}{1 - \beta u_x} &(14)\\ u'_z &= \frac{1}{\gamma}\frac{u_z}{1 - \beta u_x}\,. &(15) \end{aligned}$$

Solution

We have, through differentiation,

$$\begin{cases} dt' = \gamma\, dt - \gamma\beta\, dx \\ dx' = -\gamma\beta\, dt + \gamma\, dx \end{cases}$$

which gives

$$u'_x = \frac{dx'}{dt'} = \frac{-\gamma\beta dt + \gamma dx}{\gamma dt - \gamma\beta dx} = \frac{u_x - \beta}{1 - \beta u_x}\,.$$

Besides,

$$u'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma dt - \gamma\beta dy} = \frac{1}{\gamma}\frac{u_y}{1 - \beta u_x}\,,$$

and similarly

$$u'_z = \frac{dz'}{dt'} = \frac{dz}{\gamma dt - \gamma\beta dz} = \frac{1}{\gamma}\frac{u_z}{1 - \beta u_x}\,.$$

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2. The denote by $\vec{a} = (a_x, a_y, a_z)$ the velocity of a particle in frame R and $\vec{a}\,' = (a'_x, a'_y, a'_z)$ the corresponding velocity of in frame R'.

Show that

$$\begin{aligned} a'_x &= \frac{a_x}{\gamma^3(1 - \beta u_x)^3} &(16)\\ a'_y &= \frac{a_y}{\gamma^2(1 - \beta u_x)^2} + \frac{\beta\, u_y\, a_x}{\gamma^2(1 - \beta u_x)^3} &(17)\\ a'_z &= \frac{a_z}{\gamma^2(1 - \beta u_x)^2} + \frac{\beta\, u_z\, a_x}{\gamma^2(1 - \beta u_x)^3}\,. &(18) \end{aligned}$$

Solution

Starting from Eq. (13) we have

$$a'_x = \frac{d}{dt}\left(\frac{u_x - \beta}{1 - \beta u_x}\right)\frac{dt}{dt'}$$

and since

$$dt' = \gamma\, dt - \gamma\beta\, dx$$

one has

$$\frac{dt'}{dt} = \gamma(1 - \beta\, u_x)\,.$$

Thus

$$\begin{aligned} a'_x &= \left(\frac{a_x}{1-\beta u_x} + \frac{u_x - \beta}{(1-\beta u_x)^2}\beta a_x\right)\frac{1}{\gamma(1-\beta\, u_x)} \\ &= \frac{a_x(1-\beta^2)}{\gamma(1-\beta u_x)^3} = \frac{a_x}{\gamma^3(1-\beta u_x)^3}\,. \end{aligned}$$

Similarly, starting from Eq. (14) we have

$$\begin{aligned} a'_y &= \frac{1}{\gamma}\frac{d}{dt}\left(\frac{u_y}{1-\beta u_x}\right)\frac{1}{\gamma(1-\beta\, u_x)} = \left(\frac{a_y}{1-\beta u_x} + \frac{\beta u_y\, a_x}{(1-\beta u_x)^2}\right)\frac{1}{\gamma^2(1-\beta\, u_x)} \\ &= \frac{a_y}{\gamma^2(1-\beta u_x)^2} + \frac{\beta\, u_y\, a_x}{\gamma^2(1-\beta u_x)^3}\,. \end{aligned}$$

Exchanging the role of y and z, one finally gets Eq. (18).

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3. Starting from the 4-velocity

$$U^\mu = \frac{dX^\mu}{d\tau} = \gamma(u)(1, \vec{u}) \tag{19}$$

with $\gamma(u) = 1/\sqrt{1-u^2}$, we define the 4-vector acceleration as

$$A^\mu = \frac{dU^\mu}{d\tau}\,. \tag{20}$$

(i) Prove that

$$\frac{d\gamma(u)}{dt} = \gamma(u)^3 \vec{u}\cdot\vec{a}\,. \tag{21}$$

Solution

From

$$\gamma(u) = \frac{1}{\sqrt{1-u^2}}$$

one gets

$$\frac{d\gamma(u)}{dt} = \vec{u} \cdot \frac{d\vec{u}}{dt} \frac{1}{(1-u^2)^{3/2}} = \gamma(u)^3 \vec{u} \cdot \vec{a} \,.$$

(ii) Show that

$$A^\mu = (\gamma^4 \vec{u} \cdot \vec{a}, \gamma^2 \vec{a} + \gamma^4 (\vec{u} \cdot \vec{a}) \vec{u}) \,, \tag{22}$$

where $\gamma = \gamma(u)$.

Solution

The chain rule gives

$$\begin{aligned} A^\mu &= \frac{dU^\mu}{dt} \frac{dt}{d\tau} = \gamma(u) \frac{dU^\mu}{dt} = \gamma(u) \frac{d}{dt} \left[\gamma(u)(1, \vec{u}) \right] = \gamma(u) \left[\frac{d\gamma(u)}{dt}(1, \vec{u}) + \gamma(u) \left(0, \frac{d\vec{u}}{dt} \right) \right] \\ &= (\gamma^4 \vec{u} \cdot \vec{a}, \gamma^4 (\vec{u} \cdot \vec{a}) \vec{u} + \gamma^2 \vec{a}) \,. \end{aligned}$$

(iii) Show that A^μ is in general a space-like 4-vector.

Hint: work in the rest frame.

Solution

In the rest frame, $\vec{u} = 0$ and $\gamma = 1$, thus

$$A^\mu = (0, \vec{a}) \,,$$

so that $A^2 < 0$ (except if $\vec{a} = 0$).

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4. Orthogonality of U and A.

(i) What is the value of U^2? Deduce that U and A are orthogonal.

Solution

We have $U^2 = 1$ so that

$$0 = \frac{dU^2}{d\tau} = 2U \cdot \frac{dU}{d\tau} = U \cdot A \,.$$

(ii) Obtain this result by working in an appropriate frame.

Solution

In the rest frame, $U^\mu = (1, \vec{0})$ and $A^\mu = (0, \vec{a})$, so obviously $U \cdot A = 0$. Being true in this frame, this is valid in any frame.
Note: one can of course obtain the fact that U and A are orthogonal by directly computing $U \cdot A$:

$$
\begin{aligned}
U \cdot A &= \gamma(1, \vec{u}) \cdot (\gamma^4 \vec{u} \cdot \vec{a}, \gamma^2 \vec{a} + \gamma^4 (\vec{u} \cdot \vec{a}) \vec{u}) \\
&= \gamma^3 (\gamma^2 \vec{u} \cdot \vec{a} - \vec{u} \cdot \vec{a} - u^2 \gamma^2 (\vec{u} \cdot \vec{a})) \\
&= \gamma^3 (\gamma^2 (1 - u^2) \vec{u} \cdot \vec{a} - \vec{u} \cdot \vec{a}) = 0\,.
\end{aligned}
$$

FIN !