M1 General Physics **2022-2023**

Particles

Exam

Second session

February 9th 2023

Documents allowed

Notes:

- **The subject is deliberately long**. Solving at least one of the two problems will ensure a good mark!
- One may use the usual system of units in which $c = 1$ and $\hbar = 1$.
- Space coordinates may be freely denoted as (x, y, z) or (x^1, x^2, x^3).
- Any drawing, at any stage, is welcome, and will be rewarded!

1 Photo-production of charm

The lightest meson containing a charmed quark is the D^0. The production of a D^0 meson and of its anti-particle $\bar{D}^0$ can be done by using a beam of high energy photons which collide with protons (immobile in the reference frame of the laboratory R) according to the reaction

$$\gamma\, p \to p\, D^0\, \bar{D}^0\,. \tag{1}$$

We denote as m_p the proton mass and m_0 the D^0 mass (which is identical to the mass of the $\bar{D}^0$).

1. We seek to determine the reaction threshold, i.e. the minimum energy of the photon for which the reaction can take place. We will note E_γ the value of this energy in the laboratory reference frame.

(i) Recall the definition of the center of mass reference frame R^*.

Solution

This is the frame in which the total momentum is zero.

(ii) At threshold, the momentum in R^* of each produced particle vanishes. Write in R^* the sum of the incoming energies $E^*_\gamma + E^*_p$ as a function of m_p and m_0.

Solution

The conservation of energy implies that $E^*_\gamma + E^*_p$ should be identical to the sum of the energy of the produced particles. Since they are at rest, those energies are equal to their masses. Therefore,

$$E^*_\gamma + E^*_p = m_p + 2m_0\,.$$

(iii) Compute $(p_\gamma + p_p)^2$ in both R and R^* frames and deduce the value of E_γ as a function of m_p and m_0.

Solution

In the laboratory frame, $\vec{p}_p = 0$, therefore

$$(p_\gamma + p_p)^2 = m_p^2 + 2p_\gamma \cdot p_p = m_p^2 + 2E_\gamma E_p - 2\vec{p}_\gamma \cdot \vec{p}_p = m_p^2 + 2E_\gamma m_p = (E_\gamma^* + E_p^*)^2 = (m_p + 2m_0)^2$$

and thus

$$E_\gamma = \frac{4m_0^2 + 4m_p m_0}{2m_p} = 2m_0 + \frac{2m_0^2}{m_p}\,.$$

(iv) Compute numerically E_γ.

We give $m_p = 938$ MeV/c^2 and $m_0 = 1865$ MeV/c^2.

Solution

One gets $E_\gamma \simeq 11.15$ GeV.

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2. We want to create a beam of very energetic photons. For this we use the Compton back-scattering : a beam of electrons of 30 GeV collides head-on with a monochromatic beam of monochromatic beam of photons of wavelength $\lambda_1 = 266$ nm (a laser). The kinematics of the process is represented on the figure below in the laboratory frame R as well as in the frame R' in which the electron (bold point) is initially at rest. The incident photon is designated by 1 and the scattered photon by 2.

(i) Write the conservation of the quadri-momentum in R'. Deduce the expression of the energy E_e' of the scattered electron as a function of the energy E_1' of the incoming photon, of the energy E_2' of the scattered photon and of the mass m_e of the electron.

Solution

In the frame R', the energy of the incoming electron is m_e. The conservation of energy thus reads

$$E_1' + m_e = E_2' + E_e'$$

so that

$$E_e' = E_1' - E_2' + m_e\,.$$

(ii) Show that one has the following relation in R':

$$E'_2 = \frac{E'_1}{1 + \frac{E'_1}{m_e}(1 - \cos\theta')} \,. \tag{2}$$

Solution

Energy-momentum conservation reads

$$p'_2 - p'_1 = p_{e_i} - p_{e_f}$$

so that taking the square gives

$$(p'_2 - p'_1)^2 = -2p'_1 \cdot p'_2 = (p_{e_i} - p_{e_f})^2 = 2m_e^2 - 2m_e E'_e$$

Using the conservation of energy, see the previous question, one thus gets

$$-2E'_1 E'_2 (1 - \cos\theta') = -2m_e E'_1 + 2m_e E'_2$$

so that, as expected,

$$E'_2 = \frac{E'_1}{1 + \frac{E'_1}{m_e}(1 - \cos\theta')} \,.$$

(iii) Express the Lorentz factor γ when passing from the frame R to R', and compute its numerical value.

Compute the numerical values of E_1, E'_1 and $E'_2(\theta' = \pi)$.

We give $m_e = 0.511$ MeV/c^2 and $h = 6.626 \cdot 10^{-34}$ J.s.

Solution

One has $E_e = \gamma m_e$, so that

$$\gamma = \frac{E_e}{m_e} = \frac{30 \times 10^3}{0.511} \simeq 5.87 \cdot 10^4 \,.$$

The Lorentz transformation reads

$$\begin{aligned} E'_1 &= \gamma\, E_1 - \beta\gamma p_{1x} = \gamma\, E_1 + \beta\gamma E_1 = \gamma(1+\beta)E_1 \sim 2\gamma E_1 \\ p'_{1x} &= -\beta\gamma E_1 + \gamma p_{1x} = -\beta\gamma E_1 - \gamma E_1 = -\gamma(1 + \beta E_1) \sim -2\gamma E_1 \,. \end{aligned}$$

Besides,

$$E'_2(\theta' = \pi) = \frac{E'_1}{1 + \frac{2E'_1}{m_e}} \,.$$

Thus,

$$E_1 = \frac{h\,c}{\lambda} \simeq 7.47 \cdot 10^{-19}\,\text{J} \simeq 4.66\,\text{eV}\,,$$

$$E'_1 \simeq 548\,\text{keV}\,,$$

and

$$E'_2 \simeq 174\,\text{keV}\,.$$

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3. Backscattering

(i) Justify that $\cos\theta = -p_{x2}/E_2$, and write a similar relation in R'. Using the Lorentz transformation allowing to pass from R to R', deduce that

$$\cos\theta = \frac{\cos\theta' - \beta}{1 - \beta\cos\theta'}\,. \tag{3}$$

Solution

Since for the two photons $E_1 = \|\vec{p}_1^{\,\prime}\|$ and $E_2 = \|\vec{p}_2^{\,\prime}\|$, we have, by a simple projection on the x axis:

$$p_{x2} = -E_2\cos\theta \quad \text{and} \quad p'_{x2} = -E'_2\cos\theta'\,.$$

Besides, the Lorentz transformation reads

$$\begin{aligned} E'_2 &= \gamma E_2 - \gamma\beta p_{x2}\,, \\ p'_{x2} &= \gamma(-\beta E_2 + p_{x2})\,. \end{aligned}$$

Inserting the above expressions for p_{x2} and p'_{x2} in the second equality thus gives

$$-E'_2\cos\theta' = -\gamma E_2\left(\beta - \frac{p_{x2}}{E_2}\right) = -\gamma E_2(\beta + \cos\theta)\,.$$

The LHS of the first equality reads, using the expression of E'_2 from the Lorentz transformation as well as the expression of p_{x2}:

$$-(\gamma E_2 - \gamma\beta p_{x2})\cos\theta' = -\gamma E_2\cos\theta' - \gamma\beta E_2\cos\theta\cos\theta'\,.$$

Equating with the RHS, we get

$$-\gamma E_2\cos\theta' - \gamma\beta E_2\cos\theta\cos\theta' = -\gamma E_2(\beta + \cos\theta)$$

and thus

$$\cos\theta(1 - \beta\cos\theta) = \cos\theta' - \beta$$

which immediately leads to

$$\cos\theta = \frac{\cos\theta' - \beta}{1 - \beta\cos\theta'}\,.$$

(ii) In the SLAC setup, deduce that the photons are mainly emitted in the forward region in the laboratory frame ($\theta \sim \pi\,.$).

Solution

Since $\gamma \gg 1\,,\ \beta \to 1$ so that $\cos\theta \sim -1\,,$ i.e. $\theta \sim \pi\,.$

(iii) What is the dominant angle of emission in the frame R'?

Solution

Solving for $\cos\theta'$ gives

$$\cos\theta' = \frac{\cos\theta + \beta}{1 + \beta\cos\theta'}\,.$$

Thus, since $\beta \to 1$,

$$\cos\theta' \sim \frac{\cos\theta + 1}{1 + \cos\theta'} = 1$$

and thus again $\theta' \sim \pi\,.$

(iv) Suppose, just for the present question, that γ is arbitrary (therefore the electron may or may not be relativistic in the laboratory frame). If one detects the scattered photon at an angle $\theta = \pi$ in the laboratory frame, what would be the angle θ' in the rest frame of the electron? Comment.

Solution

Inserting $\theta = \pi$ in the previous relation gives

$$\cos\theta' = \frac{-1 + \beta}{1 - \beta} = -1$$

so that $\theta' = \pi$. This is expected from physical arguments: if the momentum of the photon has no transverse component (since $\theta = \pi$), this remains true after a longitudinal boost, so that the photon remains along the x axis. Thus, $\theta' = 0$ or $\theta' = \pi$. Besides, the boost cannot reverse its momentum by continuity with respect to the γ parameter (at $\gamma = 1$, i.e. $R = R'$, and trivially $\theta = \theta'$) so that $\theta' = \pi$.

(v) Express the energy E_2 for $\theta = \theta' = \pi$. Compute its numerical value. Comment.

Solution

From question (i), the boost implies that for $\theta = \theta' = \pi$,

$$E_2' = \gamma(1-\beta)E_2 \sim \frac{\gamma}{2}E_2$$

since

$$\gamma^2 = \frac{1}{1-\beta^2} \sim \frac{1}{\sqrt{2}\sqrt{1-\beta}}\,.$$

Thus, $E_2 \sim 2\gamma E_2' \simeq 20.4 \cdot 10^9$ eV: the amplification factor for the photon energy is enormous, since $E_2/E_1 \simeq 4.4 \cdot 10^9$!!

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4. Below is the photon energy spectrum produced at SLAC, in an experiment dedicated to charm photoproduction. Comment.

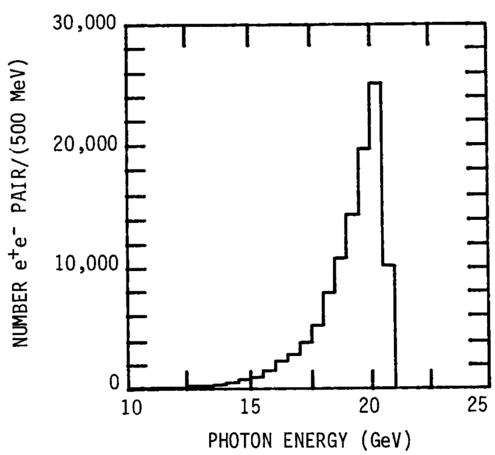

Fig. 1. Photon energy spectrum as measured by the pair spectrometer.

Figure from *AIP Conference Proceedings 113, 419 (1984).*

Solution

The energy distribution has a clear peak, corresponding to the backscattering configuration. Its value is in accordance with our result for E_2 in this configuration.

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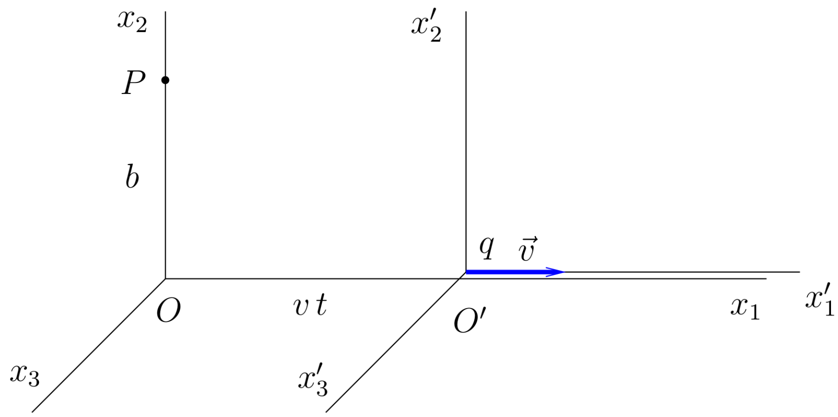
2 Field of a charge in uniform rectilinear motion

We consider a charge q in uniform rectilinear motion at the speed $\vec{v}$ in the observer's reference frame K. Let us note K' the rest frame of this charge, located at the origin O' of this one. We orientate the frames linked to K and K' so that the axes x_i and x_i' are collinear, with x_1 and x_1' pointing in the direction of the motion of the charge, and thus $\vec{v} = v$, $(v \geq 0)$. We will note t and t' the times respectively in the reference frames K and K'. We suppose that at $t = t' = 0$, the origins O and O' of the two reference frames coincide.

The observer is at a distance b from O in the reference frame K, oriented so that $\overrightarrow{OP} = b, \vec{u}_2$.



2.1 Preliminary question:

We consider two inertial reference frames K and K' so that K' is obtained from K by an arbitrary boost of velocity $\vec{v} = \vec{beta} = \vec{n}$. Let us denote $\{\vec{E}, \vec{B}\}$ and $\{\vec{E}', \vec{B}'\}$ the electromagnetic fields respectively in these two reference frames. We recall the following relations allowing us to express $\{\vec{E}', \vec{B}'\}$ using $\{\vec{E}, \vec{B}\}$:

$$\vec{E}' = (\vec{E}\cdot\vec{n})\vec{n} + \gamma\left[\vec{E} - (\vec{E}\cdot\vec{n})\vec{n}\right] + \gamma\,\vec{v}\wedge\vec{B}\,, \tag{4}$$

$$\vec{B}' = (\vec{B}\cdot\vec{n})\vec{n} + \gamma\left[\vec{B} - (\vec{B}\cdot\vec{n})\vec{n}\right] - \gamma\,\vec{v}\wedge\vec{E}\,. \tag{5}$$

Express $\{\vec{E}, \vec{B}\}$ as a function of $\{\vec{E}', \vec{B}'\}$.

Solution

On should just write the inverse transformation, which amounts to reversing the direction of $\vec{\beta}$, i.e. de $\vec{n}$ in the relations (4) and (5). On thus gets

$$\vec{E} = (\vec{E}'\cdot\vec{n})\vec{n} + \gamma\left[\vec{E}' - (\vec{E}'\cdot\vec{n})\vec{n}\right] - \gamma\,\vec{v}\wedge\vec{B}'\,,$$

$$\vec{B} = (\vec{B}'\cdot\vec{n})\vec{n} + \gamma\left[\vec{B}' - (\vec{B}'\cdot\vec{n})\vec{n}\right] + \gamma\,\vec{v}\wedge\vec{E}'\,.$$

2.2 Fields

1. Show that in K', the electromagnetic fields at point P can be written as

$$E'_1 = -\frac{qvt'}{4\pi r'^3}\,, \tag{6}$$

$$E'_2 = \frac{qb}{4\pi r'^3}\,, \tag{7}$$

$$E'_3 = 0\,, \tag{8}$$

$$\vec{B} = \vec{0}\,. \tag{9}$$

Provide the expression of r' as a function of b and t'.

Solution

The result is simply the expression of the Coulombian field of a static charge: zero magnetic field and electric field given by

$$\vec{E}' = q\frac{\vec{b} - \vec{v}t'}{\|\vec{b} - \vec{v}t'\|^3} = q\frac{\vec{b} - \vec{v}t'}{r'^3}$$

with $r' = \sqrt{b^2 + (vt')^2}$.

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2. Show that using the coordinates of K, this field also reads

$$E'_1 = -\frac{q}{4\pi}\frac{v\gamma t}{\left(b^2 + v^2\gamma^2 t^2\right)^{3/2}}, \tag{10}$$

$$E'_2 = \frac{q}{4\pi}\frac{b}{\left(b^2 + v^2\gamma^2 t^2\right)^{3/2}}. \tag{11}$$

Solution

It is enough to use the fact that $t' = \gamma(t - v\,x_1) = \gamma t$ since $x_1 = 0$ for the observer P.

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3. Show that

$$E_1 = E'_1 = -\frac{q}{4\pi}\frac{v\gamma t}{\left(b^2 + v^2\gamma^2 t^2\right)^{3/2}}, \tag{12}$$

$$E_2 = \gamma E'_2 = \frac{q}{4\pi}\frac{\gamma b}{\left(b^2 + v^2\gamma^2 t^2\right)^{3/2}}, \tag{13}$$

$$B_3 = \gamma\beta E'_2 = \beta E_2\,. \tag{14}$$

Solution

The relation which allows to express $\{\vec{E}, \vec{B}\}$ as a function of $\{\vec{E}', \vec{B}'\}$ here reads

$$\begin{aligned} \vec{E} &= E'_1\vec{u}_1 + \gamma\, E'_2\vec{u}_2 \\ \vec{B} &= \gamma\vec{\beta} \wedge \vec{E}' = \gamma v\vec{u}_1 \wedge \vec{u}_2 E'_2 = \gamma v E'_2\vec{u}_3 \end{aligned}$$

and thus

$$\begin{aligned} E_1 &= E'_1 \\ E_2 &= \gamma E'_2 \\ B_3 &= \gamma v E'_2 = \beta E_2\,. \end{aligned}$$

2.3 Non relativistic limit

4. Consider the limit $\gamma \to 1$.

i) Discuss and comment the expression of the electric field $\vec{E}$ in this limit.

Solution

We have, by confusing $\vec{r} = \vec{b} - \vec{v}t$ and $\vec{r}^{'}$ in the non relativistic limit,

$$\begin{aligned} E_1 &\sim \frac{q}{4\pi}\frac{-vt}{r^3} \\ E_2 &\sim \frac{q}{4\pi}\frac{b}{r^3} \end{aligned}$$

so that

$$\vec{E} = \frac{q}{4\pi}\frac{\vec{r}}{r^3}$$

in agreement with the expression of the Coulombian field in the absence of relativistic effect.

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ii) Same questions for the magnetic field $\vec{B}$. The result obtained will be interpreted from the point of view of the law of Biot and Savart.

Solution

In this limit, one has

$$\vec{B} \sim \frac{qvb}{4\pi r^3}\vec{u}_3 \,.$$

The law of Biot and Savart

$$\vec{B}(\vec{x}) = \int d^3y \, \frac{\vec{j}(\vec{y}) \wedge (\vec{x} - \vec{y})}{4\pi \|\vec{x} - \vec{y}\|^3}$$

here gives, since $\vec{j}(\vec{y}) = q\delta^{(3)}(\vec{y} - \vec{v}t)\vec{v}$,

$$\vec{B}(\vec{b}) = \frac{q\vec{v} \wedge (\vec{b} - \vec{v}t)}{4\pi \|\vec{b} - \vec{v}t\|^3} = \frac{qvb}{4\pi r^3}\vec{u}_3 \,,$$

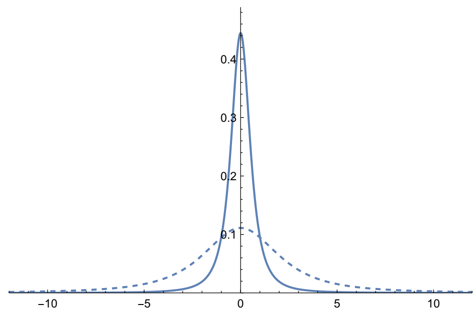
in accordance with the result obtained.

2.4 Study of relativistic effects

5. Time variation of the field transverse to the direction of motion of the particle E_2.

i) Plot the transverse field E_2 as a function of vt, for $\gamma \sim 1$ and $\gamma \gg 1$.

Solution



Plot of the field E_2 as a function of vt. In continuous line, case $\gamma = 4$, in dashed line case $\gamma = 1$. We arbitrarily set $q = 4\pi$ to fix the vertical scale.

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ii) Specify the possible extrema, and their temporal width.

Solution

The E_2 component is maximal at t=0. Its value is

$$E_{2\,max} = \frac{\gamma q}{4\pi b^2}\,.$$

For the electromagnetic field to have an appreciable amplitude compared to its maximum, it is necessary that

$$b^2 \gtrsim \gamma^2 v^2 t^2$$

so that

$$|t| \lesssim \frac{b}{\gamma v} = \Delta t\,.$$

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iii) Discuss the change in the shape of E_2 when we go from $\beta \ll 1$ to $\beta \to 1$.

Solution

From the previous question, the peak of E_2 is more pronounced and narrower the larger γ is.

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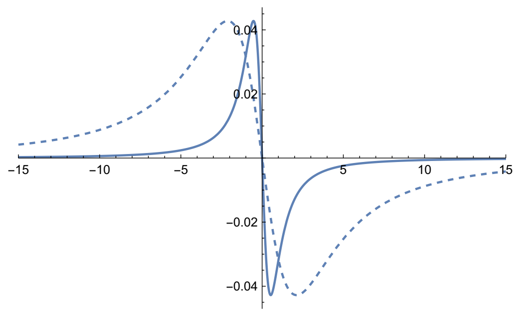
6. Time variation of the longitudinal field E_1.

i) Study and plot the longitudinal field E_1 as a function of vt, for $\gamma \sim 1$ and $\gamma \gg 1$.

Solution



Curve of the field E_1 as a function of vt. In continuous, case $\gamma = 4$, in dashed line case $\gamma = 1$. We arbitrarily set $q = 4\pi$ to fix the vertical scale.

Denotes $x = vt$ et $y = \gamma vt = \gamma x$. Then

$$E_1 = -\frac{q}{4\pi}\frac{y}{(b^2+y^2)^{3/2}}$$

and

$$\frac{d|E_1|}{|E_1|} = \frac{dy}{y} - 3\frac{ydy}{b^2+y^2}$$

vanishes for $3y^2 = b^2 + y^2$ so that $y = \pm b/\sqrt{2}$, i.e. $vt = \pm b/(\sqrt{2}\gamma)$. For these to values of y, which correspond to a maximum of $|E_1|$,

$$|E_1|_{max} = \frac{qb}{4\pi\sqrt{2}}\frac{1}{(b^2+b^2/2)^{3/2}} = \frac{q}{4\pi b^2}\frac{2}{\sqrt{27}}\,.$$

Note that these two extrema have an independent amplitude of γ.

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ii) Specify the possible extremes.

Solution

See the previous question.

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iii) Discuss the change in the shape of E_1 when we go from $\beta \ll 1$ to $\beta \to 1$.

Solution

The peaks of E_1 are all the more tightened as β is close to 1. Their amplitude does not change, contrary to the maximum of E_2.

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7. Compare the amplitude of these two fields in the $\beta \to 1$ limit

Solution

The transverse field has a maximum value typically γ times larger than the longitudinal field. In this limit, it thus dominates.

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8. i) At $t = 0$, compare the electric field transverse to the direction of motion of the particle E_2 to its non-relativistic value.

Solution

One has

$$E_2(t=0) = \frac{q}{4\pi}\frac{\gamma b}{(b^2)^{3/2}} = \frac{q}{4\pi}\frac{\gamma}{b^2} = \gamma E_{2\,non\,relativistic}\,.$$

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ii) Give an order of magnitude of the duration of the electromagnetic pulse resulting from the passage of the charged particle.

Solution

It is the temporal width $\Delta t = \frac{b}{\gamma v}$ of the peak of E_2 determined above.

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iii) Discuss the effect of the longitudinal field.

Solution

The longitudinal field E_1 varies very rapidly from a positive value (in the case where q is positive) to a negative value, and its average value is zero. This variation takes place over a time of the order of Δt. Over longer times, the effect of this field is therefore null.

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iv) For a low temporal resolution (in front of a scale to be specified), show that the $\vec{E}$ field behaves like a plane wave whose structure (polarization and direction of propagation) will be specified.

Solution

For long averaging times with respect to Δt, the perceived field is identical to that of a transversally polarized plane wave propagating along u_1: the longitudinal component has a negligible effect, and the transverse component is orthogonal to the magnetic field, both of identical amplitudes and orthogonal to $\vec{u}_1$.

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9. It is assumed that the moving charge is a particle of charge $q = ze$ and that in P is an atomic electron of charge $-e$.

i) From the above, deduce an evaluation of the impulse transferred Δp to the electron during the passage of the mobile charge. Verify that the result is independent of γ.

Solution

From the above, only the transverse field is to be considered. We have

$$\Delta p \sim zeE_2\Delta t \sim -ze^2 \frac{b}{\gamma v}\frac{1}{4\pi}\frac{\gamma}{b^2} \sim -\frac{ze^2}{4\pi bv}\,.$$

which is independent of γ.

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ii) Calculate this transferred pulse exactly.

Solution

One has

$$\begin{aligned}\int_{-\infty}^{+\infty} zeE_2(t)\,dt &= -\frac{ze^2}{4\pi vb}\int_{-\infty}^{+\infty}\frac{\gamma vt/b}{[1+(\gamma vt/b)^2]^{3/2}} = -\frac{ze^2}{4\pi vb}\int_{-\infty}^{+\infty}\frac{dx}{(1+x^2)^{3/2}}\\ &= -\frac{ze^2}{4\pi vb}\left[\frac{x}{\sqrt{1+x^2}}\right]_{-\infty}^{+\infty} = -\frac{ze^2}{2\pi vb}\,.\end{aligned}$$

Indeed,

$$\begin{aligned}\int^X \frac{dx}{(1+x^2)^{3/2}} &= -\int^{1/X}\frac{dt}{t^2}\left(1+\frac{1}{t^2}\right)^{-3/2} = -\int^{1/X}\frac{tdt}{(1+t^2)^{3/2}} = \left(1+\frac{1}{X^2}\right)^{-1/2}\\ &= \frac{X}{\sqrt{1+X^2}}\,.\end{aligned}$$
