M1 General Physics **2022-2023**

Particles

Mid-term exam

October 24th 2022

Documents allowed

Notes:
- One may use the usual system of units in which $c = 1$ and $\hbar = 1$.
- Space coordinates may be freely denoted as (x, y, z) or (x^1, x^2, x^3).
- Any drawing, at any stage, is welcome, and will be rewarded!

1 Particle decay

1.1 Particle decay in the rest frame

We consider an unstable particle of mass M, which can decay in two daughter particles of masses m_1 and m_2.
1. In the center-of-mass frame (CMS), write the 4-momentum P of the decaying particle.

Solution

$$P = (M, \vec{0})\,.$$

We denote by p the norm of 3-momentum of particles 1 and 2 in the CMS.
2. Express the sum of the two energies E_1 and E_2 of the produced particles in the CMS as a function of p.

Solution

Since $\vec{p}_1 + \vec{p}_2 = 0\,,\ p = |\vec{p}_1| = |\vec{p}_1|$ so that

$$E_1 + E_2 = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2}\,.$$

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3. Show that

$$p = \frac{1}{2M}\sqrt{[M^2 - (m_1 - m_2)^2][M^2 - (m_1 + m_2)^2]}\,. \tag{1}$$

Solution

Since

$$E_1 + E_2 = M \,,$$

and using the above expression, one thus gets

$$m_1^2 + m_2^2 + 2p^2 + 2\sqrt{m_1^2 + p^2}\sqrt{m_2^2 + p^2} = M^2$$

which leads to

$$M^2 - m_1^2 - m_2^2 - 2p^2 = +2\sqrt{m_1^2 + p^2}\sqrt{m_2^2 + p^2}$$

to that after squaring,

$$(M^2 - m_1^2 - m_2^2)^2 + 4p^4 - 4p^2(M^2 - m_1^2 - m_2^2) = 4[m_1^2 m_2^2 + m_1^2 + m_2^2)p^2 + p^4]$$

and thus

$$\begin{aligned} p &= \frac{1}{2M}\left[(m_1^2 + m_2^2 - M^2)^2 - 4m_1^2 m_2^2\right]^{1/2} \\ &= \frac{1}{2M}\left[(m_1^2 + m_2^2 - 2m_1 m_2 - M^2)(m_1^2 + m_2^2 + 2m_1 m_2 - M^2)\right]^{1/2} \\ &= \frac{1}{2M}\left[(M^2 - (m_1 - m_2)^2)(M^2 - (m_1 - m_2)^2)\right]^{1/2} . \end{aligned}$$

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4. Show that

$$M \geq m_1 + m_2 \,. \tag{2}$$

Solution

This is an immediate consequence of

$$E_1 + E_2 = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2} = M \,.$$

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5. Show that

$$E_1 = \frac{1}{2M}\left(M^2 + m_1^2 - m_2^2\right) \tag{3}$$

and

$$E_2 = \frac{1}{2M}\left(M^2 + m_2^2 - m_1^2\right) . \tag{4}$$

Solution

Inserting the above expression for p inside $E_1 = \sqrt{m_1^2 + p^2}$ we obtain

$$
\begin{aligned}
E_1 &= \left\{ m_1^2 + \frac{1}{4M^2}\left[(M^2 - (m_1 - m_2)^2)(M^2 - (m_1 - m_2)^2) \right] \right\}^{1/2} \\
&= \frac{1}{2M}\left\{ 4M^2 m_1^2 + M^4 + (m_1^2 - m_2^2)^2 - 2M^2(m_1^2 + m_2^2)) \right\}^{1/2} \\
&= \frac{1}{2M}\left\{ 2M^2 m_1^2 - 2M^2 m_2^2 + M^4 + m_1^4 + m_2^4 - 2m_1^2 m_2^2 \right\}^{1/2} \\
&= \frac{1}{2M}\left\{ (M^2 + m_1^2 - m_2^2)^2 \right\}^{1/2} = \frac{1}{2M}\left(M^2 + m_1^2 - m_2^2 \right).
\end{aligned}
$$

Similarly exchanging the role of particle 1 and 2, one gets

$$
E_2 = \frac{1}{2M}\left(M^2 + m_2^2 - m_1^2 \right).
$$

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6. Is there any preferred direction for the emitted particles? What can be said for the second particle if the first one is detected in a given direction?

Solution

The isotropy of space prevent from having a preferred direction in which the daughter particles travel (the decay is said to be isotropic). If particle 1 is detected in a given direction, then the direction of the second particle is fixed by momentum conservation: the daughter particles are traveling back-to-back in the rest frame of the mother particle.

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7. Simplify the above results in the case where the two daughter particles are equal, for instance in the decay of a neutral kaon into a pair of pions.

Solution

We have

$$
E_1 = E_2 = \frac{M}{2}
$$

and

$$
p = \frac{1}{2}\sqrt{M^2 - 4m^2}.
$$

1.2 Particle decay of an unstable particle in flight

We now consider an unstable particle in flight in the laboratory (LAB) frame. For convenience, we chose the $z-$axis as being along the direction of flight of the mother particle. We now denote $\vec{p}$ the 3-momentum of the *mother* particle, and E its energy.
We denote as $\vec{p}_{1\perp}$ and $\vec{p}_{2\perp}$ the transverse momentum (with respect to $z-$axis) of particle 1 and 2 respectively, and p_{1z} and p_{2z} their longitudinal momentum, so that the momenta of daughter particles read

$$\begin{aligned} p_1 &= (E_1, \vec{p}_{1\perp}, p_{1z})\,, &(5)\\ p_2 &= (E_2, \vec{p}_{2\perp}, p_{2z})\,. &(6)\end{aligned}$$

1. Write P, the 4-momentum of the mother particle, and compare $\vec{p}_{1\perp}$ and $\vec{p}_{2\perp}$, and p, p_{1z} and p_{2z}.

Solution

One has

$$P = (E, 0, 0, p)\,.$$

Since $P = p_1 + p_2$, we thus have $p = p_{1z} + p_{2z}$ and $\vec{p}_\perp \equiv \vec{p}_{1\perp} = -\vec{p}_{2\perp}$.

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2. We now use asterisks for momenta in the CMS.
Write precisely the boost from the CMS to the LAB frame for the mother particle.

Solution

The active boost from the CMS to the LAB frame reads

$$\left\{\begin{array}{lcl} p^0 &=& \gamma p^{0*} + \gamma\beta p^{z*} \\ p^z &=& \gamma\beta p^{0*} + \gamma p^{z*} \\ \vec{p}_\perp &=& \vec{p}_\perp^{\,*} = 0\,. \end{array}\right.$$

Since $p^{0*} = M$ and $p^0 = E$, and since $p^{z*} = 0$ and $p^z = p$, we have $\gamma = \frac{E}{M}$ and $\beta = \frac{p}{\gamma M} = \frac{p}{E}$.

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3. Deduce the boost for particle 1 and 2.

Solution

$$\left\{\begin{array}{lcl} E_1 &=& \gamma(E_1^* + \beta p_{1z}^*) \\ p_{1z} &=& \gamma(p_{1z}^* + \beta E_1^*) \\ \vec{p}_{1\perp} &=& \vec{p}_{1\perp}^{\,*} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} E_2 &=& \gamma(E_2^* + \beta p_{2z}^*) \\ p_{2z} &=& \gamma(p_{2z}^* + \beta E_2^*) \\ \vec{p}_{2\perp} &=& \vec{p}_{2\perp}^{\,*}\,. \end{array}\right.$$

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4. Show that the LAB angle θ_1 that daughter particle 1 makes with the direction of flight of the mother particle in a two-body decay is related to the CMS angle θ_1^* by the following equation

$$\tan\theta_1 = \frac{\sin\theta_1^*}{\gamma(\beta/\beta_1^* + \cos\theta_1^*)} \tag{7}$$

where β is the LAB velocity of the mother particle and β_1^* the CMS velocity of the daughter particle.

Solution

We have

$$\tan\theta_1 = \frac{p_{1\perp}}{p_{1z}} = \frac{p_{1\perp}^*}{\gamma p_{1z}^* + \gamma\beta E_1^*} = \frac{p_{1\perp}^*}{p_1^*}\frac{1}{\gamma\frac{p_{1z}^*}{p_1^*} + \gamma\beta\frac{E_1^*}{p_1^*}} = \frac{\sin\theta_1^*}{\gamma(\beta/\beta_1^* + \cos\theta_1^*)}$$

since $\beta_1^* = \frac{p_1^*}{E_1^*}$ and $\cos\theta_1^* = \frac{p_{1z}^*}{p_1^*}$.

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2 Train in a tunnel

Preliminary:
Show that in a boost from a frame F in which an object is at rest, to an arbitrary frame F', lengths along the direction of the boost gets contracted by a factor γ, while transverse distances are unaffected.
Hint: consider a rod at rest in a frame F, parallel to the axis of the boost. Determine its length in the boosted frame F', i.e. measured *at one and the same time* t' in this frame F'.

Solution

In a Lorentz boost, say of axis z,

$$\begin{cases} t &= \gamma t' + \gamma\beta z' \\ z &= \gamma\beta t' + \gamma z' \end{cases}$$

while the transverse coordinates are preserved. At time t', the two extremities of the rod parallel to the z-axis satisfy

$$z_1 = \gamma\beta t' + \gamma z_1' \qquad \text{and} \qquad z_2 = \gamma\beta t' + \gamma z_2'$$

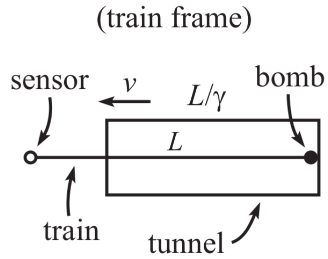
i.e. $z_2 - z_1 = \gamma(z_2' - z_1')$ and thus $\Delta z' = \frac{1}{\gamma}\Delta z$.

A train and a tunnel both have proper lengths L. The train moves toward the tunnel at speed v. A bomb is located at the front of the train. The bomb is designed to explode when the front of the train passes the far end of the tunnel. A deactivation sensor is located at the back of the train. When the back of the train passes the near end of the tunnel, the sensor tells the bomb to disarm itself. Does the bomb explode? To answer to this question, we will consider two different frames, with two answers which should be obviously identical.

1. Easiest discussion: First consider the frame of the train and conclude.

Solution

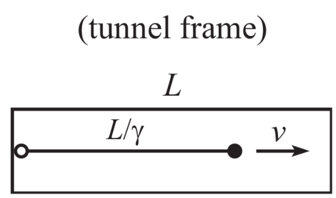
In the frame of the train, the train has length L, and the tunnel speeds past it. The tunnel is length-contracted down to L/γ. Therefore, the far end of the tunnel passes the front of the train before the near end passes the back, so the bomb explodes.



2. Naively paradoxical result: Consider now the frame of the tunnel. Conclude.
Hint: Escape the paradox by taking into account the time of propagation of any signal.

Solution

We can, however, also look at things in the frame of the tunnel. Here the tunnel has length L, and the train is length-contracted down to L/γ. Therefore, the deactivation device gets triggered before the front of the train passes the far end of the tunnel, so you might think that the bomb does not explode. We appear to have a paradox. The resolution to this paradox is that the deactivation device cannot instantaneously tell the bomb to deactivate itself. It takes a finite time for the signal to travel the length of the train from the sensor to the bomb. And it turns out that this transmission time makes it impossible for the deactivation signal to get to the bomb before the bomb gets to the far end of the tunnel, no matter how fast the train is moving. Let us show this in detail.



The signal has the best chance of winning this "race" if it has speed c, so let's assume this is the case. Now, the signal gets to the bomb before the bomb gets to the far end of the tunnel if and only if a light pulse emitted from the near end of the tunnel (at the instant the back of the train goes by) reaches the far end of the tunnel before the front of the train does. The former takes a time L/c. The latter takes a time $L(1-1/\gamma)/v$, because the front of the train is already a distance L/γ through the tunnel. So if the bomb is not to explode, the following constraint should hold:

$$
\begin{aligned}
\frac{L}{c} &< \frac{L}{v}\left(1-\frac{1}{\gamma}\right) \\
\Leftrightarrow \beta &< 1-\sqrt{1-\sqrt{1-\beta^2}} \\
\Leftrightarrow \sqrt{1-\beta^2} &< 1-\beta \\
\Leftrightarrow \sqrt{1+\beta} &< \sqrt{1-\beta}\,.
\end{aligned}
$$

Obviously, this never occurs. This shows that signal always arrives too late, and the bomb always explodes.

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3 Impossibility of certain processes

1. Show that the process $\gamma \to e^- e^+$ is impossible in the vacuum.

Solution

One should have

$$\vec{p}_\gamma = \vec{p}_{e^-} + \vec{p}_{e^+}$$

and

$$E_\gamma = E_{e^-} + E_{e^+} = \sqrt{p_{e^-}^2 + m_e^2} + \sqrt{p_{e^+}^2 + m_e^2}$$

with $p_{e^-} = |\vec{p}_{e^-}|$ and $p_{e^+} = |\vec{p}_{e^+}|$. Besides, the constraint

$$E_\gamma = |\vec{p}_\gamma| = |\vec{p}_{e^-} + \vec{p}_{e^+}| \leq |\vec{p}_{e^-}| + |\vec{p}_{e^+}|$$

should hold, i.e.

$$\sqrt{p_{e^-}^2 + m_e^2} + \sqrt{p_{e^+}^2 + m_e^2} \leq p_{e^-} + p_{e^+}$$

which is obviously impossible since $m_e \neq 0$.

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2. Show that this reaction is possible in the vicinity of a heavy nucleus and calculate the threshold energy for this reaction.

Solution

Conservation of energy and momentum can now be fulfilled thanks to the possibility that the nucleus A gets a recoil from the process. The threshold energy is obtained by moving to the γA rest frame: one should have $E^* = E^*_\gamma + E^*_A \geq 2m_e + M_A$. In the rest frame of the initial nucleus, this reads

$$\begin{aligned} E^{*2} &= (p_\gamma + p_A)^2 = M^2 + 2p_\gamma \cdot p_A = M^2 + 2E_\gamma M \\ &\geq (2m_e + M_A)^2 = 4m_e^2 + 4m_e M_A + M^2 \end{aligned}$$

i.e.

$$E_\gamma \geq 2m_e + \frac{2m_e^2}{M_A}\,.$$

Neglecting the mass of the electron with the mass of the nucleus, this simplifies to $E_\gamma \geq 2m_e$.

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3. Show that it is impossible for a free and isolated electron to absorb a photon.

Solution

Let us denote $(E_1, \vec{p}_1)$ and $(E_2, \vec{p}_2)$ the 4-momentum of the initial and final electron. We have

$$m^2 = E_1^2 - p_1^2 = E_2^2 - p_2^2 = (E_1 + E_\gamma)^2 - (\vec{p}_1 + \vec{p}_\gamma)^2 = E_1^2 + 2E_1 E_\gamma - p_1^2 - 2\vec{p}_1 \cdot \vec{p}_\gamma\,.$$

Thus,

$$E_1 E_\gamma - p_1^2 - \vec{p}_1 \cdot \vec{p}_\gamma = 0$$

i.e., introducing the angle θ between the initial photon and the initial electron,

$$E_\gamma \left[\sqrt{m_e^2 + p_1^2} - p_1 \cos\theta\right] = 0\,.$$

A non trivial solution $E_\gamma \neq 0$ would thus lead to

$$\cos\theta = \frac{\sqrt{m_e^2 + p_1^2}}{p_1} > 1$$

for $m_e \neq 0$, which is obviously absurd.
