M1 General Physics **2023-2024**

Particles

Mid-term exam

November 7th 2023

Documents allowed

Notes:
- **Exercises are independent**.
- The questions or exercises with a * are more difficult. Any serious trial will be rewarded.
- One may use the usual system of units in which $c = 1$ and $\hbar = 1$.
- Space coordinates may be freely denoted as (x, y, z) or (x^1, x^2, x^3).
- Any drawing, at any stage, is welcome, and will be rewarded!

1 A photon travels at the speed of a photon!

A photon moves at an angle θ with respect to the x' axis in frame S'. Frame S' moves at speed v with respect to frame S (along the x' axis).

1. Calculate the components of the photon's velocity in S, and verify that its speed is c, as it should!

Solution

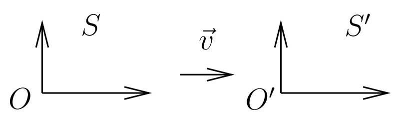
One has

$$\left\{\begin{array}{lcl} V'_x &=& c\,\cos\theta \\ V'_y &=& c\,\sin\theta \\ V'_z &=& 0 \end{array}\right.$$

and thus

$$\left\{\begin{array}{lcl} dx' &=& V'_x\,dt' = c\,\cos\theta \\ dy' &=& V'_y\,dt' = c\,\sin\theta \\ dz' &=& 0\,. \end{array}\right.$$

Besides, the boost between S and S'



reads

$$\left\{\begin{array}{lcl} ct &=& \gamma\,ct' + \gamma\beta x' \\ x &=& \gamma\beta\,ct' + \gamma x' \\ y &=& y' \\ z &=& z'\,. \end{array}\right.$$

i.e.

$$\begin{cases} cdt &= \gamma\, cdt' + \gamma\beta dx' = \gamma\, cdt' + \gamma\beta \cos\theta\, cdt' \\ dx &= \gamma\beta\, cdt' + \gamma dx' = \gamma\beta\, cdt' + \gamma\cos\theta\, cdt' \\ dy &= dy' = \sin\theta\, cdt' \\ dz &= dz' = 0\,. \end{cases}$$

so that

$$\begin{cases} dt &= \gamma(1+\beta\cos\theta)dt' \\ dx &= \gamma(\beta+\cos\theta)cdt' \\ dy &= \sin\theta\, cdt' \\ dz &= 0\,. \end{cases}$$

and finally

$$\begin{cases} V_x &= \frac{dx}{dt} = \frac{\beta+\cos\theta}{1+\beta\cos\theta}c \\ V_y &= \frac{dy}{dt} = \frac{dx}{dt} = \frac{\sin\theta}{\gamma(1+\beta\cos\theta)}c \\ V_z &= 0\,. \end{cases}$$

One easily checks that

$$\begin{aligned} v^2 &= V_x^2 + V_y^2 = \frac{c^2}{(1+\beta\cos\theta)^2}\left[(\beta+\cos\theta)^2 + \sin^2\theta\right] \\ &= \frac{c^2}{(1+\beta\cos\theta)^2}(\beta^2 + 2\beta\cos\theta + \cos^2\theta + \sin^2\theta) = c^2 \end{aligned}$$

as expected.

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2. Express the frequency ν' of the photon in frame S' in terms of its frequency ν in frame S, the angle θ and the speed v.

Solution

Let us consider the boost between S and S'. It reads for the 4-momentum of the photon:

$$p^0 = \gamma\, p^{0'} + \gamma\,\beta\, p'_{\|} = \gamma p^{0'}(1+\beta\cos\theta)$$

with $p^0 = h\nu$ and $p^{0'} = h\nu'$ so that

$$\nu = \nu'\gamma(1+\beta\cos\theta)\,,$$

or equivalently

$$\nu' = \frac{\nu}{\gamma(1+\beta\cos\theta)}\,.$$

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3. Discuss the various limits of interest.

Solution

At fixed β:

$$\begin{aligned}
\theta &= 0: \qquad \nu' = \nu'_{min} = \frac{\nu'}{\gamma(1+\beta)} = \sqrt{\frac{1-\beta}{1+\beta}}: \qquad \text{maximal red-shift effect,}\\
\theta &= \frac{\pi}{2}: \qquad \nu' = \frac{\nu}{\gamma} = \nu\sqrt{1-\beta^2} \le \nu: \qquad \text{red-shift effect,}\\
\theta &= \pi: \qquad \nu' = \nu'_{max} = \frac{\nu}{\gamma(1-\beta)} = \nu\sqrt{\frac{1+\beta}{1-\beta}} \ge \nu: \qquad \text{maximal blue-shift effect.}
\end{aligned}$$

2 Photoproduction of a particle

A photon, of frequency ν, collides with a particle of mass M and produces in the final state a particle of mass M', close to M, and a particle of mass m, which is said to be photoproduced by the reaction.

1. What is the minimum frequency required to cause this type of reaction? Application to the photoproduction of charged pions by reaction with protons:

$$\gamma\, p \to n\, \pi^+. \tag{1}$$

Solution

The center of mass energy should be at least equal to the sum of the masses of the produced particles, i.e. $E^* \ge (M'+m)c^2$. Since

$$E^{*2} = (p_\gamma + p_M)^2 c^2 = 2 p_\gamma \cdot p_M c^2 + M^2 c^4$$

where we have used the mass-shell condition for the photon and for the particle of mass M, and assuming that the initial particle is at rest, i.e. $p_\gamma \cdot p_M = E_\gamma M = h\nu M$, we get

$$\nu = \frac{\left[(M'+m)^2 - M^2\right] c^2}{2hM}\,.$$

which gives $\nu \simeq 3.66 \times 10^{22}$ Hz.

2. Simplifies this result in the case $M' = M$, and discuss the case of the photoproduction of π^0, as

$$\gamma\, p \to p\, \pi^0. \tag{2}$$

Compare the minimal frequencies in the two above cases.

Solution

When $M = M'$,

$$\nu = \frac{[2Mm + m^2]\, c^2}{2hM}\,.$$

which gives $\nu \simeq 3.50 \times 10^{22}$ Hz .

Data: $M_p c^2 = 938.272$ MeV, $M_n c^2 = 939.565$ MeV, $m_{\pi^+} c^2 = 139.57$ MeV, $m_{\pi^0} c^2 = 134.976$ MeV, $h = 6.626 \times 10^{-34}$ J·s, 1 eV $= 1.602 \times 10^{-19}$ J.

3 Particles ψ and ψ'

The e^+e^- collision ring SPEAR (SLAC) was of one of the two experiments (the other one being at BNL) which led to the discovery of ψ (now called J/ψ) particle, and thus of charm quark, announced on November 11th 1974. Indeed, ψ and ψ' can be produced through the reactions

$$e^+\ e^- \to \psi \tag{3}$$

and

$$e^+\ e^- \to \psi'\,. \tag{4}$$

The experimental configuration is such that the two beams collide head-on, and carry the same energy.

1. What should be the kinetic energies T_e of e^+ and e^- to allow for these two reactions? Show that the speed of electron is close to the speed of light, with a relative precision better than 10^{-6}.

Solution

The threshold energy is $m_\psi c^2$, and since each particle carries half of this, after subtracting the rest energy one gets that threshold energy is given for each incoming lepton by

$$T_e(\text{thres.}) = \frac{mc^2}{2} - m_e c^2.$$

One gets for reaction (3), $T_e = 1548$ MeV and for reaction (4), $T_e = 1842$ MeV. Besides, we have $E = \gamma m_e c^2$ with $\gamma = (1-\beta^2)^{-1/2}$. Since β is very close to unity, we have $\gamma^2 \simeq 1/(2(1-\beta))$ and thus

$$1 - \beta \sim \left(\frac{m_e c^2}{T_e}\right)^2\,.$$

This leads to $1-\beta \simeq 5.45 \times 10^{-8}$ for the first reaction and $1-\beta \simeq 3.85 \times 10^{-8}$ for the second.

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2. The particle ψ' can decay at rest according to the reaction

$$\psi' \to \psi\ n\pi^+\ n\pi^-. \tag{5}$$

What is the maximal value of n, the number of pairs of charged pions? In the following, the work in the rest frame of the ψ', and we consider the case $n = 1$.

Solution

Each pair costs an energy of $2m_\pi c^2$ and thus $n < (m_{\psi'} - m_\psi)/(2m_\pi)$. Thus $n = 1$ or 2.

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3. What are the kinetic energies T_+ and T_- of the π^+ and π^- mesons when the particle ψ of the process (5) is produced at rest?

Solution

We have $T_+ + T_- + 2m_\pi c^2 + m_\psi = m_{\psi'} c^2$ with $T_+ = T_-$ since the total 3-momentum of the pair should be 0, which means that each pion carries the same momentum, and thus has the same energy, thus the same kinetic energy. Therefore, $T_+ = T_- = (m_{\psi'} - m_\psi - 2m_\pi)c^2/2 \simeq 155$ MeV.

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4. When ψ is not at rest, provide a relationship between T_+, T_-, the kinetic energy T of the particle ψ and the relative angle θ between the flight directions of the two pions.

Solution

We have now $\vec{p}_\psi = \vec{p}_{\pi^+} + \vec{p}_{\pi^-}$ and thus, using the notation $p = |\vec{p}|$, $p_\psi^2 = p_{\pi^+}^2 + p_{\pi^-}^2 + 2p_{\pi^+}p_{\pi^-}\cos\theta$. Besides, for any particle, $m^2c^4 = E^2 - p^2c^2 = (mc^2 + T)^2 - p^2c^2$ and thus $p^2c^2 = 2mc^2T + T^2 = T(T + 2mc^2)$. Inserting this relation for ψ, π^+ and π^- leads to

$$T(T + 2m_\psi c^2) = T_+(T_+ + 2m_\pi c^2) + T_-(T_- + 2m_\pi c^2) + 2\cos\theta\,[T_+T_-(T_+ + 2m_\pi c^2)(T_- + 2m_\pi c^2)]^{1/2}.$$

Data: $m_e c^2 = 0.511$ MeV, $m_\psi c^2 = 3097$ MeV, $m_{\psi'} c^2 = 3686$ MeV, $m_{\pi^\pm} c^2 = 139.57$ MeV.

4 Maximum energy carried by a particle

We consider either the decay process

$$A \to D_1 + D_2 + D_3 \cdots D_N \tag{6}$$

or the scattering process

$$A + B \to D_1 + D_2 + D_3 \cdots D_N\,. \tag{7}$$

We first work in the center-of-mass frame R, denoting the total energy as E. Our aim is to compute the maximal energy $E_i(max)$ of particle i in this frame. To simplify notations, we denote $p_i = |\vec{p}_i|$. Without loss of generality, we first consider particle 1. We will come back to the general case in question 8.

1. Express $\vec{p}_1$ and E_1 in terms of $\vec{p}_2, \cdots \vec{p}_N$ and $E_2, \cdots E_N$ respectively.

Solution

One has

$$\begin{aligned}\vec{p}_1 &= -(\vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_N)\,,\\ E_1 &= E - (E_2 + E_3 + \cdots + E_N)\,.\end{aligned}$$

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2. We denote $E_r = E_2 + \cdots E_N$. Show that p_1 is maximal when E_r is minimal.

Solution

The relation $p_1^2 = E_1^2 - m_1^2$ obviously imposes that $|\vec{p}_1|$ is maximal when E_1 is maximal, and thus when E_r is minimal.

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3. We now consider the center-of-mass frame R' of the $N-1$ particles $D_2, \cdots D_N$. Introducing the Lorentz factor γ between the two frames, relate E_r and E'_r, the energy of the $N-1$ particles respectively in frame R and frame R'.

Solution

One has $E_r = \gamma E'_r$.

Deduce that E_r is minimal when E'_r is itself minimal, and that this implies that each particle should be at rest in frame R'.

Solution

From the previous question, we deduce that E_r is minimum when E'_r is minimum. But E'_r is minimal when each particle $i \in \{2, \cdots N\}$ is at rest in R'.

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4. Explain why in such a configuration, every particle is flying in the same direction in frame R.

Solution

When each particle $i \in \{2, \cdots N\}$ is at rest in R', it means that getting back to frame R, the corresponding boost will make each particle flying parallel, in the direction of the boost from R to R', with a velocity equal to the one of R' with respect to R. Particle 1 is moving in the opposite direction.

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5. Show then that in this configuration,

$$\forall i \in \{2, \cdots N\}, \frac{\vec{p}_i}{m_i} = \frac{\vec{p}_2}{m_2}\,. \tag{8}$$

Solution

From the previous study, we know that in this configuration, $\vec{v}_2 = \vec{v}_3 = \cdots = \vec{v}_N$. Besides,

$$\forall i \in \{2, \cdots N\}, \frac{\vec{v}_{R'/R}}{c^2} = \frac{\vec{v}_i}{c^2} = \frac{\vec{p}_i}{E_i} = \frac{\vec{p}_i}{\gamma(v_i) m_i c^2}$$

where $\gamma(v_i) = \gamma(v_{R'/R})$, so that

$$\forall i \in \{2, \cdots N\}, \frac{\vec{p}_i}{m_i} = \frac{\vec{p}_2}{m_2}\,.$$

6. Prove that

$$p_1(max) = p_2 \frac{M}{m_2}\,, \tag{9}$$

with $M = m_2 + \cdots m_N$.

Solution

From $\vec{p}_1 = -(\vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_N)$ and the fact that every particle $i \in \{2, \cdots N\}$ move in the same direction, we deduce that

$$p_1 = \sum_{i=2}^{N} p_i = p_2 \left(1 + \sum_{i=3}^{N} \frac{m_i}{m_2}\right) = p_2 \frac{M}{m_2}\,.$$

7. Show that

$$E = \sqrt{p_1(max)^2 c^2 + m_1^2 c^4} + \sqrt{p_1(max)^2 c^2 + M^2 c^4} \tag{10}$$

Solution

One has

$$E = E_1 + \cdots E_N = \sqrt{p_1(max)^2 c^2 + m_1^2 c^4} + \sum_{i=2}^{N} \sqrt{p_i^2 c^2 + m_i^2 c^4}\,.$$

The second term in the rhs of the previous equation reads, using the fact that (see question 7.)

$$\forall i \in \{2, \cdots N\}, \frac{p_i}{m_i} = \frac{p_2}{m_2} = \frac{p_1(max)}{M}\,,$$

$$\begin{aligned}
\sum_{i=2}^{N} \sqrt{p_i^2 c^2 + m_i^2 c^4} &= \sum_{i=2}^{N} \sqrt{\frac{p_i^2}{m_i^2} m_i^2 c^2 + m_i^2 c^4} = \sum_{i=2}^{N} m_i \sqrt{\frac{p_1(max)^2}{M^2} c^2 + c^4} = M \sqrt{\frac{p_1(max)^2}{M^2} c^2 + c^4} \\
&= \sqrt{p_1(max)^2 c^2 + M^2 c^4}
\end{aligned}$$

from which we get the desired result.

8. Derive finally the desired result, after some algebra and an obvious relabeling:

$$cp_i(max) = \frac{\sqrt{(E + m_i c^2 + M_i c^2)(E - m_i c^2 - M_i c^2)(E + m_i c^2 - M_i c^2)(E - m_i c^2 + M_i c^2)}}{2E} \qquad (11)$$

with $M_i = m_1 + m_2 + \cdots m_{i-1} + m_{i+1} + \cdots m_N$.

Solution

First, squaring the previous result leads to

$$E^2 = p_1(max)^2c^2 + m_1^2c^4 + 2\sqrt{p_1(max)^2c^2 + m_1^2c^4}\sqrt{p_1(max)^2c^2 + M^2c^4} + p_1(max)^2c^2 + M^2c^4$$

and thus

$$E^2 - 2p_1(max)^2c^2 - m_1^2c^4 - M^2c^4 = 2\sqrt{p_1(max)^2c^2 + m_1^2c^4}\sqrt{p_1(max)^2c^2 + M^2c^4} \, .$$

Squaring again, one gets on the one hand

$$\begin{aligned} & (E^2 - 2p_1(max)^2c^2 - m_1^2c^4 - M^2c^4)^2 \\ = \; & E^4 + 4p_1(max)^4c^4 + m_1^4c^8 + M^4c^8 - 4E^2p_1(max)^2c^2 - 2E^2m_1^2c^4 - 2E^2M^2c^4 \\ & +4p_1(max)^2m_1^2c^6 + 4p_1(max)^2M^2c^6 + 2m_1^2M^2c^8 \end{aligned}$$

and on the other hand

$$\begin{aligned} & 4(p_1(max)^2c^2 + m_1^2c^4)(p_1(max)^2c^2 + M^2c^4) \\ = \; & 4p_1(max)^4c^4 + 4p_1(max)^2m_1^2c^6 + 4p_1(max)^2M^2c^6 + 4m_1^2M^2c^8 \, . \end{aligned}$$

Equating these two expressions and after simplification, one obtains

$$E^4 + m_1^4c^8 + M^4c^8 - 4E^2p_1(max)^2c^2 - 2E^2m_1^2c^4 - 2E^2M^2c^4 = 2m_1^2M^2c^8$$

and thus

$$\begin{aligned} p_1(max)^2c^2 &= \frac{E^4 + m_1^4c^8 + M^4c^8 - 2E^2m_1^2c^4 - 2E^2M^2c^4 - 2m_1^2M^2c^8}{4E^2} \\ &= \frac{E^4 - 2(m_1^2 + M^2)c^4E^2 + (m_1^2 - M^2)^2c^8}{4E^2} \\ &= \frac{(E^2 - (m_1 - M)^2c^2)(E^2 + (m_1 + M)^2c^2)}{4E^2} \\ &= \frac{(E + m_1c^2 + Mc^2)(E - m_1c^2 - Mc^2)(E + m_1c^2 - Mc^2)(E - m_1c^2 + Mc^2)}{4E^2} \end{aligned}$$

so that, after the relabeling $m_1 \to m_i$ and $M \to M_i$, we finally obtain for an arbitrary particle i,

$$p_i(max)^2c^2 = \frac{(E + m_ic^2 + M_ic^2)(E - m_ic^2 - M_ic^2)(E + m_ic^2 - M_ic^2)(E - m_ic^2 + M_ic^2)}{4E^2}$$

which leads to the desired result.
Here is another way of performing the calculation: starting from

$$E = \sqrt{p_1(max)^2c^2 + m_1^2c^4} + \sqrt{p_1(max)^2c^2 + M^2c^4}$$

one has

$$E - \sqrt{p_1(max)^2c^2 + M^2c^4} = \sqrt{p_1(max)^2c^2 + m_1^2c^4}$$

and thus

$$E^2 + p_1(max)^2c^2 + M^2c^4 - 2E\sqrt{p_1(max)^2c^2 + M^2c^4} = p_1(max)^2c^2 + m_1^2c^4\,,$$

and thus

$$E^2 + M^2c^4 - m_1^2c^4 = 2E\sqrt{p_1(max)^2c^2 + M^2c^4}\,,$$

which leads to

$$p_1(max)^2c^2 + M^2c^4 = \frac{(E^2 + M^2c^4 - m_1^2c^4)^2}{4E^2}\,,$$

so that

$$\begin{aligned}
p_1(max)^2c^2 &= \frac{(E^2 + M^2c^4 - m_1^2c^4)^2 - 4E^2M^2c^4}{4E^2} \\
&= \frac{(E^2 + M^2c^4 - m_1^2c^4 - 2EMc^2)(E^2 + M^2c^4 - m_1^2c^4 + 2EMc^2}{4E^2} \\
&= \frac{[(E - Mc^2)^2 - m_1^2c^4][(E + Mc^2)^2 - m_1^2c^4]}{4E^2} \\
&= \frac{[E - Mc^2 - m_1c^2][E - Mc^2 + m_1c^2][E + Mc^2 - m_1c^2][E + Mc^2 + m_1c^2]}{4E^2}\,.
\end{aligned}$$

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5 A passing stick

A stick of length L moves past you at speed v. There is a time interval between the front end coinciding with you and the back end coinciding with you.

1. What is this time interval in your frame? Obtain the result by working in your frame.

Solution

The stick has length L/γ in your frame. It moves with speed v. Therefore, the time taken in your frame to cover the distance L/γ is $L/(\gamma v)$

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2. Same question, but now working in the stick's frame. You should of course get the same result as in 1.!

Solution

From the point of view of the stick, you fly at speed v. In its frame, the stick has length L, so that the time elapsed in the stick frame is L/v. Besides, using time dilation, time in the stick frame is seen to flow a factor of γ longer with respect to your frame, in which you measure the time we are interested in. Thus, the time in your frame is equal to $L/(\gamma v)$. This is hopefully in agreement with question 1.

3(***!). What is this time interval in the stick's frame? Obtain the result by working in your frame.

Solution

The rear clock on the stick shows an additional elapsed time of with respect to the front clock of

$$\frac{L(c+v)}{2c} - \frac{L(c-v)}{2c} = \frac{Lv}{c^2} \tag{12}$$

due the difference of length propagation of the light (in order that the light hits simultaneously each end of the stick in your frame. Besides, more time will elapse on the rear clock by the time it reaches you. The time in your frame is $L/(\gamma v)$ taking into account the length of the stick in your frame, which is L/γ. Thus, the total time difference is

$$\frac{Lv}{c^2} + \frac{L}{\gamma^2 v} = \frac{L}{v}\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right) = \frac{L}{v}\left(\frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}\right) = \frac{L}{v}\,. \tag{13}$$

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4. Same question, but now working in the stick's frame.

Solution

Same reasoning as at the beginning of 2.: from the point of view of the stick, you fly at speed v. In its frame, the stick has length L, so that the time elapsed in the stick frame is L/v.
