M1 General Physics

Particles

Cross-sections

1 Invariant one-particle phase-space

1. Show that

$$\frac{d^3\vec{p}}{(2\pi)^3 2E(|\vec{p}\,|)} = \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2 - m^2)\theta(p_0) \tag{1}$$

where $E = E(|\vec{p}\,|) = \sqrt{\vec{p}^{\,2} + m^2}$, and conclude about the Lorentz invariance of this one-particle phase-space.
Hint: use the fact that

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|} \tag{2}$$

where x_i are the simple roots of $f(x)$.

2. Write $d^3\vec{p}$ in terms of $p = |\vec{p}|$ (beware to this rather standard notation: p here should not be confused with the 4-momentum!) and of the elementary solid angle $d^2\Omega$.

3. Write $d^2\Omega$ in spherical coordinates.

2 Phase space in the center-of-mass frame

We consider the $2 \to 2$ process $A(p_A)\,B(p_B) \to C(p_C)\,D(p_D)$, where A, B, C, D are particles of mass respectively equal to m_A, m_B, m_C, m_D. Our aim is to simplify the expression of the phase space

$$d(P.S) = (2\pi)^4\delta^{(4)}(p_A + p_B - p_C - p_D)\frac{d^3p_C}{(2\pi)^3 2E_C}\frac{d^3p_D}{(2\pi)^3 2E_D} \tag{3}$$

in the center-of-mass frame. We denote $p_C = |\vec{p}_C|$ and $p_D = |\vec{p}_D|$. One may use the Mandelstam variable $s = (p_A + p_B)^2$. In the center-of-mass frame, we denote $p_f^* = p_C$.

1. Show that in the center-of-mass frame,

$$d(P.S) = \frac{1}{4\pi^2}\delta^{(3)}(\vec{p}_C + \vec{p}_D)\,\delta(E_C(p_C) + E_D(p_D) - \sqrt{s})\,\frac{d^3p_C}{2E_C(p_C)}\frac{d^3p_D}{2E_D(p_D)}\,, \tag{4}$$

and give the expressions of $E_C(p_C)$ and $E_D(p_D)$.

2. Show finally that

$$d(P.S) = \frac{1}{4\pi^2} \frac{p_f^*}{4\sqrt{s}} d^2\Omega \,. \tag{5}$$

Hint: one may use Eq. (2).

3 Study of the "spinless" electron-muon scattering

Consider "spinless" electron-muon scattering. Denote θ the scattering angle in the center-of-mass system (c.m.s), i.e. the angle between the outgoing and incoming electron (or muon) momentum. One may use the notation of section 1, with $m_e = m_A = m_C$ and $m_\mu = m_B = m_D$.

1. Write the expression of the scattering amplitude $\mathcal{M}$.

2. We denote $s = (p_A + p_B)^2$. In the c.m.s., write the equations satisfied by $\vec{p}_A$, $\vec{p}_B$, $\vec{p}_C$, $\vec{p}_D$ and E_A, E_B, E_C, E_D. Deduce an equation satisfied by $|\vec{p}_A|$ and $|\vec{p}_C|$ and conclude about their relative magnitude.
Then, write the energy and space components of p_A, p_B, p_C, p_D in terms of $s = (p_A + p_B)^2$ and of E_A, E_B, $\vec{p}_A$ and $\vec{p}_C$.

3. Give the expression of q^2 as a function of θ and $|\vec{p}_A|$. Then, write q^2 in terms of $s = (p_A + p_B)^2$, m_A, m_B. One may use the obtained expression for $|\vec{p}_A|$ in the 2024 mid-term exam, or directly solve the equation satisfied by $|\vec{p}_A|$ in question 2.

4. Express the numerator of $\mathcal{M}$ as a function of E_A, E_B, $\vec{p}_A^{\,2}$ and $\cos\theta$. Write E_A and E_B in terms of s, m_A, m_B (one may rely on results obtained in the 2024 mid-term exam) and show finally that

$$\mathcal{M} = e^2 \left[\frac{3 + \cos\theta}{1 - \cos\theta} + \frac{C}{1 - cos\theta} \right] \tag{6}$$

where C is a function of s, m_A, m_B which vanishes in the high-energy limit.

5. Prove finally that the differential cross-section reads

$$\left.\frac{d\sigma}{d\Omega}\right|_{cms} = \frac{\alpha^2}{4s} \left(\frac{3 + C + \cos\theta}{1 - \cos\theta} \right)^2 , \tag{7}$$

where $\alpha = e^2/(4\pi)$ is the fine-structure constant.