M1 General Physics

Particles

Cross-sections

1 Invariant one-particle phase-space

1. Show that

$$\frac{d^3\vec{p}}{(2\pi)^3 2E(|\vec{p}\,|)} = \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m^2)\theta(p_0) \tag{1}$$

where $E = E(|\vec{p}\,|) = \sqrt{\vec{p}^{\,2}+m^2}$, and conclude about the Lorentz invariance of this one-particle phase-space.
Hint: use the fact that

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} \tag{2}$$

where x_i are the simple roots of $f(x)$.

Solution

One should just write

$$\delta(p^2-m^2)\theta(p_0) = \delta(p_0^2-\vec{p}^{\,2}-m^2)\theta(p_0)$$

The equation $p_0^2-\vec{p}^{\,2}-m^2=0$ has two roots $p_0 = \pm E(|\vec{p}\,|) = \sqrt{\vec{p}^{\,2}+m^2}$, but the positive energy constraint $\theta(p_0)$ selects the positive one. Since

$$\frac{d}{dp_0}(p_0^2-\vec{p}^{\,2}-m^2)(p_0=E) = 2E\,,$$

we thus have

$$\delta(p^2-m^2)\theta(p_0) = \frac{\delta(p_0-E)}{2E}$$

q.e.d. The Lorentz invariance is obvious from the R.H.S of Eq. (1).

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2. Write $d^3\vec{p}$ in terms of $p=|\vec{p}|$ (beware to this rather standard notation: p here should not be confused with the 4-momentum!) and of the elementary solid angle $d^2\Omega$.

Solution

$$d^3p = p^2\,dp\,d^2\Omega\,.$$

3. Write $d^2\Omega$ in spherical coordinates.

Solution

$$d^2\Omega = \sin\theta d\theta d\phi\,.$$

2 Phase space in the center-of-mass frame

We consider the $2 \to 2$ process $A(p_A)\,B(p_B) \to C(p_C)\,D(p_D)$, where A, B, C, D are particles of mass respectively equal to m_A, m_B, m_C, m_D. Our aim is to simplify the expression of the phase space

$$d(P.S) = (2\pi)^4\delta^{(4)}(p_A + p_B - p_C - p_D)\frac{d^3p_C}{(2\pi)^3 2E_C}\frac{d^3p_D}{(2\pi)^3 2E_D} \tag{3}$$

in the center-of-mass frame. We denote $p_C = |\vec{p}_C|$ and $p_D = |\vec{p}_D|$. One may use the Mandelstam variable $s = (p_A + p_B)^2$. In the center-of-mass frame, we denote $p_f^* = p_C$.

1. Show that in the center-of-mass frame,

$$d(P.S) = \frac{1}{4\pi^2}\delta^{(3)}(\vec{p}_C + \vec{p}_D)\,\delta(E_C(p_C) + E_D(p_D) - \sqrt{s})\,\frac{d^3p_C}{2E_C(p_C)}\frac{d^3p_D}{2E_D(p_D)}\,, \tag{4}$$

and give the expressions of $E_C(p_C)$ and $E_D(p_D)$.

Solution

In the center-of-mass frame, we have $\vec{p}_A + \vec{p}_B = 0$ and thus $s = (p_A + p_B)^2 = (E_A + E_B)^2$. The 4-momenta conservation then reads $\vec{p}_C + \vec{p}_D = 0$, so that $p_C = p_D$, and $E_C + E_D = E_A + E_B = \sqrt{s}$.
Besides, $E_C = \sqrt{m_C^2 + p_C^2}$ and $E_D = \sqrt{m_D^2 + p_D^2} = \sqrt{m_D^2 + p_C^2}$. Thus,

$$\delta^{(4)}(p_A + p_B - p_C - p_D) \quad = \quad \delta^{(3)}(\vec{p}_C + \vec{p}_D)\,\delta(E_C(p_C) + E_D(p_C) - \sqrt{s})\,.$$

2. Show finally that

$$d(P.S) = \frac{1}{4\pi^2}\frac{p_f^*}{4\sqrt{s}}d^2\Omega\,. \tag{5}$$

Hint: one may use Eq. (2).

Solution

We have

$$d(P.S) = \frac{1}{4\pi^2}\delta(f(p_C))\,\frac{p_C^2\,dp_C\,d^2\Omega}{2E_C(p_C)\,2E_D(p_C)}\,,$$

with $f(p_C) = \sqrt{m_C^2 + p_C^2} + \sqrt{m_D^2 + p_C^2} - \sqrt{s}$. We denote p_f^* the positive root of $f(p_C)$. Since

$$f'(p_C) = \frac{p_C}{\sqrt{m_C^2 + p_C^2}} + \frac{p_C}{\sqrt{m_D^2 + p_C^2}} = \frac{p_C(\sqrt{m_C^2 + p_C^2} + \sqrt{m_D^2 + p_C^2})}{\sqrt{m_C^2 + p_C^2}\sqrt{m_D^2 + p_C^2}} = \frac{p_C\sqrt{s}}{E_C(p_C)\,E_D(p_C)},$$

we thus get

$$\delta(f(p_C))\,\frac{p_C^2\,dp_C}{2E_C(p_C)\,2E_D(p_C)} = \delta(p_C - p_f^*)\,\frac{1}{|f'(p_f^*)|}\,\frac{p_C^2\,dp_C}{2E_C(p_C)\,2E_D(p_C)} = \frac{p_f^*}{4\sqrt{s}}$$

and thus

$$d(P.S) = \frac{1}{4\pi^2}\frac{p_f^*}{4\sqrt{s}}d^2\Omega\,.$$

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3 Study of the “spinless” electron-muon scattering

Consider “spinless” electron-muon scattering. Denote θ the scattering angle in the center-of-mass system (c.m.s), i.e. the angle between the outgoing and incoming electron (or muon) momentum. One may use the notation of section 1, with $m_e = m_A = m_C$ and $m_\mu = m_B = m_D$.

1. Write the expression of the scattering amplitude $\mathcal{M}$.

Solution

The scattering amplitude $\mathcal{M}$ reads

$$i\mathcal{M} = [ie(p_A + p_C)^\mu]\left[-i\frac{g_{\mu\nu}}{q^2}\right][ie(p_B + p_D)^\mu]\,,$$

where $q^2 = t = (p_A - p_C)^2$, and thus

$$\mathcal{M} = e^2(p_A + p_C)\cdot(p_B + p_D)\frac{1}{q^2}\,.$$

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2. We denote $s = (p_A + p_B)^2$. In the c.m.s., write the equations satisfied by $\vec{p}_A$, $\vec{p}_B$, $\vec{p}_C$, $\vec{p}_D$ and E_A, E_B, E_C, E_D. Deduce an equation satisfied by $|\vec{p}_A|$ and $|\vec{p}_C|$ and conclude about their relative magnitude.
Then, write the energy and space components of p_A, p_B, p_C, p_D in terms of $s = (p_A + p_B)^2$ and of E_A, E_B, $\vec{p}_A$ and $\vec{p}_C$.

Solution

In the c.m.s, one has $\vec{p}_A + \vec{p}_B = 0$ and $\vec{p}_C + \vec{p}_D = 0$, and thus $\sqrt{s} = E_A + E_B = E_C + E_D$. First, this implies that

$$|\vec{p}_A| = |\vec{p}_B| = \sqrt{E_A^2 - m_A^2} = \sqrt{E_B^2 - m_B^2} \text{ and } |\vec{p}_C| = |\vec{p}_D| = \sqrt{E_C^2 - m_A^2} = \sqrt{E_D^2 - m_B^2}\,.$$

Second, $\sqrt{s} = E_A + E_B = E_C + E_D$ reads

$$\sqrt{m_A^2 + \vec{p}_A^{\,2}} + \sqrt{m_B^2 + \vec{p}_A^{\,2}} = \sqrt{m_A^2 + \vec{p}_C^{\,2}} + \sqrt{m_B^2 + \vec{p}_C^{\,2}}$$

and thus $|\vec{p}_A| = |\vec{p}_C|$. We can thus write

$$\begin{aligned} p_A &= (E_A, \vec{p}_A) \\ p_B &= (E_B, -\vec{p}_A) \\ p_C &= (E_A, \vec{p}_C) \\ p_D &= (E_B, -\vec{p}_C)\,. \end{aligned}$$

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3. Give the expression of q^2 as a function of θ and $|\vec{p}_A|$. Then, write q^2 in terms of $s = (p_A + p_B)^2$, m_A, m_B. One may use the obtained expression for $|\vec{p}_A|$ in the 2024 mid-term exam, or directly solve the equation satisfied by $|\vec{p}_A|$ in question 2.

Solution

We have

$$\begin{aligned} q^2 = (p_A - p_C)^2 &= p_A^2 + p_C^2 - 2p_A \cdot p_C = 2m_A^2 - 2E_A E_C + 2|\vec{p}_A||\vec{p}_C|\cos\theta \\ &= 2m_A^2 - 2E_A^2 + 2|\vec{p}_A|^2\cos\theta = -2|\vec{p}_A|^2(1-\cos\theta) \\ &= -\frac{[s-(m_A-m_B)^2][s-(m_A+m_B)^2]}{2s}(1-\cos\theta) \end{aligned}$$

where we have used the fact that in the c.ms.,

$$|\vec{p}_A|^2 = \frac{[s-(m_A-m_B)^2][s-(m_A+m_B)^2]}{4s},$$

obtained either by solving

$$\sqrt{m_A^2 + \vec{p}_A^{\,2}} + \sqrt{m_B^2 + \vec{p}_A^{\,2}} = \sqrt{s}$$

or using Eq. (13) of the mid-term exam of November 2024.

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4. Express the numerator of $\mathcal{M}$ as a function of E_A, E_B, $\vec{p}_A^{\,2}$ and $\cos\theta$. Write E_A and E_B in terms of s, m_A, m_B (one may rely on results obtained in the 2024 mid-term exam) and show finally that

$$\mathcal{M} = e^2\left[\frac{3+\cos\theta}{1-\cos\theta} + \frac{C}{1-cos\theta}\right] \tag{6}$$

where C is a function of s, m_A, m_B which vanishes in the high-energy limit.

Solution

$$(p_A+p_C)\cdot(p_B+p_D) = (2E_A, \vec{p}_A+\vec{p}_C)\cdot(2E_B, -\vec{p}_A-\vec{p}_C) = 4E_AE_B + 2\vec{p}_A^{\,2}(1+\cos\theta)\,.$$

We have, either computing $E_A = \sqrt{m_A^2+\vec{p}_A^{\,2}}$ and $E_B = \sqrt{m_B^2+\vec{p}_A^{\,2}}$ from the obtained expression for $\vec{p}_A^{\,2}$, or using Eq. (16) of the 2024 mid-term exam,

$$E_A = \frac{s+m_A^2-m_B^2}{2\sqrt{s}} \quad \text{and} \quad E_B = \frac{s+m_B^2-m_A^2}{2\sqrt{s}}$$

and thus

$$\begin{aligned}&(p_A+p_C)\cdot(p_B+p_D)\\ &= \frac{2(s+m_A^2-m_B^2)(s+m_B^2-m_A^2) + (s-(m_A-m_B)^2)(s-(m_A+m_B)^2)(1+\cos\theta)}{2s}\end{aligned}$$

so that

$$\begin{aligned}\mathcal{M} &= e^2\left[2\frac{s^2-(m_A-m_B)^2(m_A+m_B)^2}{(s-(m_A-m_B)^2)(s-(m_A+m_B)^2)}\frac{1}{1-\cos\theta} + \frac{1+\cos\theta}{1-\cos\theta}\right]\\ &= e^2\left[\frac{3+\cos\theta}{1-\cos\theta} + 4\frac{s(m_A^2+m_B^2)-(m_A^2-m_B^2)}{(s-(m_A-m_B)^2)(s-(m_A+m_B)^2)}\frac{1}{1-\cos\theta}\right]\end{aligned}$$

5. Prove finally that the differential cross-section reads

$$\left.\frac{d\sigma}{d\Omega}\right|_{cms} = \frac{\alpha^2}{4s}\left(\frac{3+C+\cos\theta}{1-\cos\theta}\right)^2, \tag{7}$$

where $\alpha = e^2/(4\pi)$ is the fine-structure constant.

Solution

We know that in the c.ms., $p_i^* = |\vec{p}_A| = |\vec{p}_C| = p_f^*$ so that the differential cross-section reads

$$\left.\frac{d\sigma}{d\Omega}\right|_{cms} = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}|\mathcal{M}|^2 = \frac{e^4}{64\pi^2 s}\left(\frac{3+C+\cos\theta}{1-\cos\theta}\right)^2 = \frac{\alpha^2}{4s}\left(\frac{3+C+\cos\theta}{1-\cos\theta}\right)^2.$$