M1 General Physics

Particles

Relativistic kinematics

Exercise 1: Relative velocity

Consider two particles of momenta p^μ and q^μ. Provide a covariant expression of their relative velocity, defined in a frame in which one of the two is at rest. It will be useful to compute $p \cdot q$ in such a frame.

Solution

In the frame in which one of the two particles is at rest, for example in the frame where particle 1 is at rest, with $p = (m_1, \vec{0})$ and $q = (\frac{m_2}{\sqrt{1-v^2}}, \vec{q})$, one gets

$$p \cdot q = \frac{m_1 m_2}{\sqrt{1-v^2}}$$

where v is the relative velocity. Thus

$$\boxed{v = \sqrt{1 - \frac{m_1^2 m_2^2}{(p\cdot q)^2}}\,.}$$

with $m_1^2 = p^2$ and $m_2^2 = q^2$.
Since $p{\cdot}q$ is a Lorentz scalar, this expression as the same expression in any frame, thus allowing to compute the relative velocity without any explicit use of a Lorentz transformation.

Exercise 2: Fixed target experiments versus collider experiments

1. We send a particle of mass m, of kinetic energy K (total energy minus energy at rest, i.e $K = E - m$), on another identical particle at rest (so-called *fixed target* experiment). Compute the energy in the center-of-mass frame.

Solution

In the center-of-mass frame, $\vec{p_1} + \vec{p_2} = 0$ and thus

$$E^* = \sqrt{(p_1+p_2)^2} = \sqrt{p_1^2 + p_2^2 + 2p_1\cdot p_2} = \sqrt{2m^2 + 2mE} = \sqrt{4m^2 + 2mK}$$

where we have used the fact that $p_1 \cdot p_2 = mE$ in the frame where particle 1 is at rest, and the relation $K = E - m$.

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2. We accelerate two particles of opposite momenta, of mass m and of kinetic energy K^* (collider mode, the symbol $*$ is used to refer to the center-of-mass frame). What should be the value of K in the fixed target experiment of question 1. in order to reach the same energy in the center-of-mass frame? Discuss the limits $K^* \ll m$ and $K^* \gg m$. Explain why most of the modern experiments in high-energy physics are made using colliders (LEP at CERN, HERA at Hamburg, Tevatron at Fermilab near Chicago, SLC in Stanford, LHC, future EIC (2030), ILC e^+e^- projects, FCC-ee and FCC-pp).
The RHIC collider, at Brookhaven (near New-York) collides two beams of heavy ions, for example Au-Au with 200 GeV per nucleon. Compute the energy K of a similar fixed target experiment. One should compare this result to the highest possible energy available at SPS, which was a fixed target experiment (beams of 160 GeV per nucleon).

Solution

We now have for the total energy $E^* = 2(m + K^*)$. In order to reach the same energy in the center-of-mass frame using a fixed target experiment, one should thus satisfy

$$2(m + K^*) = \sqrt{4m^2 + 2mK}$$

that is

$$8mK^* + 4m^2 + 4K^{*2} = 4m^2 + 2mK$$

and thus

$$\boxed{K = 4K^* + \tfrac{2K^{*2}}{m} = 4K^*[1 + \tfrac{K^*}{2m}]\,.}$$

Non relativistic limit: $K^* \ll m$
The velocity of the incident particle should be 2 times larger in the fixed target experiment, and therefore carry a kinetic energy 4 times larger.

Relativistic limit: $K^* \gg m$
The enhancement factor $4 + \frac{K^*}{2m}$ then increases dramatically.
In fixed target experiments, one cannot acheive very high values of E^*.
At RHIC, $K^* = 200$ GeV/nucleon, and thus

$$K = 4 \times 200 \times A \left(1 + \frac{200}{2}\right)$$

that is $\frac{K}{A} \simeq 80.8\ TeV$/nucleon ! Even at the LHC, with a nominal energy of 7 TeV per charge, i.e. for ${}^{82}_{207}$Pb, an energy of 2.76 TeV/nucleon, this cannot be achieved. Nevertheless, fixed target experiment are very useful in various situations, where the center-of-mass energy is not the key parameter:
- whenever polarization of the target is required (it is much more difficult to polarize a beam of proton then a fixed target for example)
- when the density of the target can have a very sizable effect on the magnitude of cross-sections (it is much easier to have a dense target than a high intensity beam).

3. One collides two beams of ultra-relativistic particles ($E \gg m$) in the opposite direction, of energies E_1 and E_2. Compute the center-of-mass energy.
The HERA collider (1992-2007) was colliding a beam of protons of 800 GeV with a beam of electrons of 30 GeV. Compute the center-of-mass energy.

Solution

Neglecting masses, we have, assuming the collision axis is the $z-$axis, $p_1 = (E_1, 0, 0, E_1)$ and $p_1 = (E_2, 0, 0, -E_2)$ so that

$$E^* = \sqrt{(p_1 + p_2)^2} = \sqrt{2p_1 \cdot p_2} = \sqrt{2E_1 E_2 + 2E_1 E_2} = 2\sqrt{E_1 E_2}\,.$$

At HERA, this gives $E^* \simeq 310$ GeV.

Exercise 3: Photoproduction of pions

Consider the reaction $\gamma p \to \pi^0 p$, where p means a proton, of mass $M = 939$ MeV, γ is a photon and π^0 a neutral pion, of mass $m \simeq 135$ MeV. We denote respectively P, $k = (k^0, \vec{k})$ and $q = (q^0, \vec{q})$ the four-momenta of the incoming proton, of the incoming photon and of the emitted pion.

1. We assume that the proton is initially at rest. Compute, literally and then numerically, the minimal energy of the photon in order that the reaction $\gamma p \to \pi^0 p$ would be possible.

Solution

Let us first study the threshold of an arbitrary reaction. In the center-of-mass frame, summing over the produced particles labeled by i, we get on the one hand

$$(\sum p_i)^2 = (\sum E_i)^2 = E^{*2}\,.$$

On the other hand, since $E_i = \sqrt{\vec{p}_i^{\,2} + m_i^2} \geqslant m_i$ so that $E^{*2} \geqslant (\sum m_i)^2$ and thus

$$\boxed{E^* \geqslant \sum m_i\,,}$$

with an equality when $\forall i,\ \vec{p}_i^{\,2} = 0$. The minimal rest frame total energy (named as threshold energy) is thus equal to $\sum m_i$.
Here, $E^* \geqslant M + m$. Besides, $E^{*2} = (P + k)^2 = M^2 + 2P \cdot k$. In the frame where the proton is at rest, we have $P \cdot k = M\, k^0$ and thus, combining the two above results,

$$M^2 + 2Mk^0 \geqslant (M + m)^2 = M^2 + 2Mm + m^2$$

and finally

$$\boxed{k^0 \geqslant m + \tfrac{m^2}{2M}\,.}$$

Note that in the limit of an infinite mass M, one recover the obvious constraint $k^0 \geqslant m$ obtained when producing a proton at rest.
This gives numerically $k^0 \geqslant 144.7$ MeV.

2. Compute the energy of the incident photon as a function of the emitted pion energy q^0 and of the angle θ of the pion momenta with respect to the incident photon.

Solution

Conservation of energy-momentum reads $k + P = q + P_f$ where P_f is the momentum of final proton. Thus $k + P - q = P_f$ and squaring gives $(k + P - q)^2 = P_f^2 = M^2$ so that $k^2 + (P - q)^2 + 2k \cdot (P - q) = M^2$ and thus, since $k^2 = 0$ and $(P - q)^2 = M^2 - 2P \cdot q + m^2$,

$$2k \cdot (P - q) = 2P \cdot q - m^2$$

In the frame where the proton is at rest, this reads

$$2k \cdot (P - q) = 2k^0(M - q^0) + 2k^0|\vec{q}\,| \cos\theta = 2Mq^0 - m^2$$

and finally

$$\boxed{k^0 = \frac{q^0 - m^2/(2M)}{1 - (q^0 - |\vec{q}\,| \cos\theta)/M}}$$

with $|\vec{q}\,| = \sqrt{q_0^2 - m^2}$.

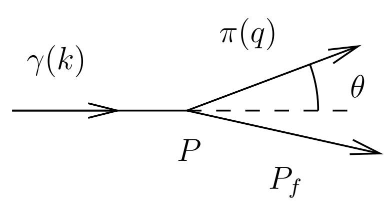

Figure 1: Scattering in the rest frame of the proton.

3. Simplify the previous results in the limit where m and k^0 are very small with respect to M. Comment on the result.

Solution

The relationship

$$2k^0(M - q^0) + 2k^0|\vec{q}| \cos\theta = 2Mq^0 - m^2$$

now reduces to $2Mk^0 = 2Mq^0$ i.e. $k^0 = q^0$: all the energy of the photon is transfered to the pion. The threshold energy is m.

4. This reaction plays a crucial role in the physics of high energy cosmic rays: indeed, the Universe is bathed with photon (the cosmological radiation at 3K) and protons of very high energy can scatter with these photons and produce pions, thus slowing them down. Consider thus the scattering of a proton of energy E and a photon of energy $k^0 = 10^{-3}$ eV of opposite direction (this is the order of magnitude of the 3K radiation). Compute the minimal value of E in order that the reaction would be possible. This cut is known under the name of Greisen-Zatsepin-Kuzmin (1966). One of the enigma of current studies of cosmic rays is the fact that there are signals of cosmic rays of higher energies.

Solution

The condition $E^* > M + m$ reads, since $E^{*2} = (P+k)^2 = M^2 + 2P \cdot k$,

$$M^2 + 2P \cdot k \geqslant M^2 + 2Mm + m^2$$

i.e. $2P \cdot k \geqslant 2Mm + m^2$. The proton and the photon are going in opposite directions, therefore, denoting $p = |\vec{p}|$,

$$P \cdot k = Ek^0 - \vec{p} \cdot \vec{k} = Ek^0 + pk^0$$

and the condition thus reads

$$E + p \geqslant \frac{2Mm + m^2}{2k^0}.$$

In the limit $k^0 \ll m$ this leads to $E + p \gg M$ and thus $p \simeq E$. We finally get

$$E \geqslant \frac{2Mm + m^2}{4k^0} \simeq 7.10^{19}\,\text{eV}$$

If the collision is not head-on, introducing the angle θ between $-\vec{k}$ and $\vec{p}$, see Fig. 2,

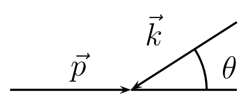

Figure 2: A non head-on collision.

on gets

$$P \cdot k = Ek^0 + pk^0 \cos\theta$$

and the threshold condition $P \cdot k \geqslant mM + m^2/2$ reads

$$Ek^0 + pk^0 \cos\theta \geqslant mM + \frac{m^2}{2}$$

with $p \sim E$ so that we get

$$E \geqslant \frac{2Mm + m^2}{2(1+\cos\theta)k^0} \geqslant E^{\text{threshold}}_{\theta=0}.$$

The proton scatters a large number of photons of the cosmological bath, which are randomly distributed, therefore in principle any proton of energy larger than the above cut will reduce its energy, until it becomes smaller that this cut. Such a strong suppression has been seen by Auger and HiRes experiments. However, it turns out that there exist a few number of events with an energy above this cut. This might be explained, although the situation is not yet clarified, by the fact that the mean path is of the order of 6 Mpc (one parsec is roughly 3.3 light-year; this mean path accounts for both the cross-section and the density of cosmological photons), so that proton traveling over distances larger than 50 Mpc will travel enough to reduce their energy below this cut. Thus, any proton emitted at a smaller distance, for example from a source inside the Milky Way (which has a stellar size of the order of 200 Mly), may not have time to loose its energy through scattering over the cosmological photon bath. Another explanation is the fact that ultra-energetic cosmic rays can be heavier elements than protons, so that Greisen-Zatsepin-Kuzmin cut does not apply.

Exercise 4: Compton effect

Compton scattering is the elastic scattering of a photon of momentum $\vec{k}$ with an electron of mass m and of momentum $\vec{p}$. We denote by $\vec{k}'$ the momentum of the scattered photon, $E = \sqrt{\vec{p}^2 + m^2}$ the energy of the incoming electron, and θ the angle between $\vec{k}$ and $\vec{p}$.

1. Check that $P \cdot K = (P + K) \cdot K'$.
Infer that the energy of the scattered photon equals

$$|\vec{k}'| = \frac{E - p\cos\theta}{E + |\vec{k}| - \hat{\vec{k}'} \cdot (\vec{p} + \vec{k})} |\vec{k}| \tag{1}$$

where we denote as $\hat{k}' \equiv \vec{k}'/|\vec{k}'|$ the direction of $\vec{k}'$.

Solution

The kinematical variables and the angles are illustrated in Fig. 3. Conservation of energy-

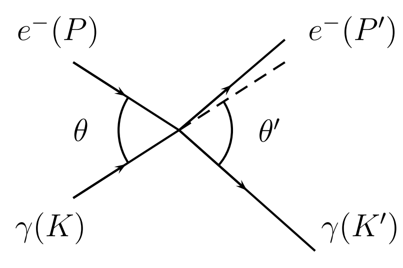

Figure 3: Compton scattering.

momentum reads $P + K - K' = P'$ which, using the mass-shell condition for the out-going electron leads to

$$\begin{aligned} (P + K - K')^2 &= P'^2 = m^2 \\ &= P^2 + K^2 + K'^2 + 2P \cdot K - 2(P + K) \cdot K' \end{aligned}$$

with $P^2 = m^2, K^2 = K'^2 = 0$, which leads to $P \cdot K = (P+K) \cdot K'$.
Denoting $P = (E, \vec{p})$, $K = (k^0, \vec{k})$, $K' = (k^{0'}, \vec{k}')$ and $k = |\vec{k}| = k^0, k' = |\vec{k}'| = k^{0'}$, we thus get

$$Ek - \vec{p}.\vec{k} = Ek' + kk' - (\vec{p} + \vec{k}) \cdot \vec{k}'$$

then

$$(E - p\cos\theta)k = (E+k)k' - (\vec{p} + \vec{k}) \cdot \vec{k}' = [(E+k) - (\vec{p} + \vec{k}) \cdot \hat{\vec{k}}']k'$$

and finally

$$\boxed{k^{0'} = \frac{E - p\cos\theta}{E + k^0 - (\vec{p} + \vec{k}) \cdot \hat{\vec{k}}'} k^0 \,.}$$

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2. We assume that the electron is at rest. Compute the wave length λ' of the scattered photon as a function of the wave length λ of the incoming photon, of the Compton wave length defined as $\lambda_C = h/mc$, and of the angle θ' between $\vec{k}$ and $\vec{k}'$.

Solution

The electron is assumed to be at rest: $\vec{p} = \vec{0}$ and $E = m$. Thus

$$k^{0'} = \frac{m}{(m + k^0) - \vec{k} \cdot \hat{\vec{k}}'} k^0 = \frac{k^0}{1 + \frac{k^0}{m} - \frac{k^0}{m}\cos\theta'} \,.$$

From $\lambda = \frac{h}{k}$ and $\lambda_C = \frac{h}{m}$ we get

$$\frac{1}{\lambda'} = \frac{1}{\lambda} \frac{1}{1 + \frac{\lambda_C}{\lambda} - \frac{\lambda_C}{\lambda}\cos\theta'} = \frac{1}{\lambda + \lambda_C(1 - \cos\theta')}$$

and thus

$$\boxed{\lambda' = \lambda + \lambda_C(1 - \cos\theta') \,.}$$

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3. In the more general case where $\vec{p} \neq \vec{0}$, deduce from equation (1) the maximal energy of the diffused photon for a given θ.

Solution

$k^{0'}$ is maximal when, for fixed θ and k^0, the denominator in the expression of $k^{0'}$ is minimal, i.e. $(\vec{p} + \vec{k}) \cdot \hat{\vec{k}}'$ is maximal, which means $\vec{k}'$ and $\vec{p} + \vec{k}$ collinear and pointing in the same direction. We thus get

$$\boxed{k^{0'}_{\max} = \frac{E - p\cos\theta}{E + k^0 - |\vec{p} + \vec{k}|} k^0 \,.}$$

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4. We assume the electron to be ultrarelativistic. Check that in the limit of a low energy photon,

$$|\vec{k}'|_{\max} = \left(\frac{2E\sin(\theta/2)}{m}\right)^2 |\vec{k}|\,. \tag{2}$$

Specify the condition on $|\vec{k}|$ for this equation to be valid. Check the validity of the various approximations and calculate $|\vec{k}'|_{\max}$ in the case of a laser beam of energy 1 eV interacting with an electron beam of energy $E = 6$ GeV, 12 GeV since 2016, in opposite direction. These are typical value of the Compton polarimeter used at Thomas Jefferson National Accelerator Facility (JLab), located in Virginia (USA). Such a device allows, using a polarized laser, to measure the polarization of the electron beam. It also allows to produce high energy photons with a partial polarization (using the maximal energy of the produced photon corresponding to the backscattering regime).

Solution

In the limit of an ultra-relativistic electron and a low energy photon, one can make the approximation $k \ll p$, E and $m \ll p$, E, so that

$$k^{0\prime}_{\max} \sim \frac{E(1-\cos\theta)}{E-p} k^0\,.$$

Since

$$E = \sqrt{\vec{p}^2 + m^2} \sim p + \frac{m^2}{2p} \quad \text{and thus} \quad E - p \sim \frac{m^2}{2E}$$

one gets

$$k^{0\prime}_{\max} \sim \frac{E(1-\cos\theta)}{m^2/2p} k^0$$

and thus

$$\boxed{k^{0\prime}_{\max} \sim \left(\frac{2E\sin\theta/2}{m}\right)^2 k^0\,.}$$

This is a very appealing result: the energy of the out-going photon scales like the square of the energy of the incoming electron. It is maximal in the backward configuration in which the initial photon scatters head-on with the electron ($\theta = \pi$) and the out-going photon is emitted backward ($\theta' = \pi$ since here $\vec{p} + \vec{k} \sim \vec{p}$ points in the same direction as $\vec{k}'$). Selecting those photons, one can produce a very energetic beam of photon from a low energy laser source, pumping the energy of the incoming beam of electrons.
Note that the high-energy approximation which we have used is valid as soon as $m^2/(2E) \gg k^0$ which at JLab gives, for $E = 6$ GeV, $k^0 \ll 22$ eV and for $E = 12$ GeV, $k^0 \ll 11$ eV, producing beams of photons of maximal energies respectively equal to 0.55 GeV and 2.2 GeV.

Exercise 5: Electron scattering

In this exercise, we consider the scattering, either elastic or inelastic, of an ultrarelativistic electron (its mass will be neglected) on a target. We denote by $K = (k, \vec{k})$ and $K' = (k', \vec{k'})$ the four-momenta of the electron before and after the collision, and $P = (E, \vec{p})$ and P' those of the target, which has a mass M. We assume that $\vec{k}$ and $\vec{p}$ are collinear and of opposite directions.

If the scattering is elastic, the energy k' of the outgoing electron is a function of the angle between $\vec{k}$ and $\vec{k'}$ (see the previous exercise). This angle will be denoted here by θ. In the more general case of an inelastic scattering, k' and θ are independent variables.

1. It turns out to be convenient to use, instead of k' and θ, the Lorentz invariant variables Q^2 and $P \cdot q$, where $q \equiv K - K'$. Compute these two quantities as functions of k', θ, E and k. What is the sign of q^2? Traditionally, one denotes $Q^2 = -q^2$.

Solution

The kinematics is illustrated in Fig. 4. From $q = K - K'$ we have

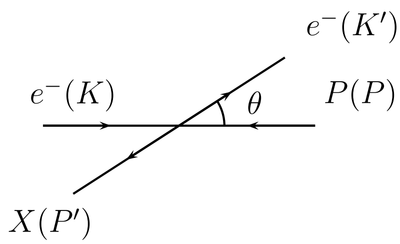

Figure 4: Head-on electron-proton scattering, where X denotes the proton remnants.

$$q^2 = (K - K')^2 = K^2 + K^{'2} - 2K \cdot K' = -2K \cdot K' = -2kk'(1 - \cos\theta) \leqslant 0\,,$$

so that we naturally denote $Q^2 = -q^2 \geqslant 0$, and

$$P \cdot q = P \cdot K - P \cdot K' = Ek + kp - Ek' - pk' \cos\theta$$

where we denote $p = |\vec{p}|$.

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2. Consider a scattering which transforms the target into a particle of mass M', with $M' \geqslant M$. What is the relationship between Q^2 and $P \cdot q$? Infer the sign of $P \cdot q$. Whenever M' is close to M, such a scattering is named quasi-elastic.

Solution

From $K - K' + P = P'$ we get

$$(K - K' + P)^2 = M^{'2} = q^2 + 2q \cdot P + P^2 = q^2 + 2q \cdot P + M^2$$

so that $q^2 + 2q \cdot P = M^{'2} - M^2 \geqslant 0$ and thus, since $q^2 < 0$, $P \cdot q \geqslant 0\,.$

3. We assume that the electron faces an elastic scattering off a part of a target, characterized by a four-momentum xP, with $0 \leq x \leq 1$, the rest of the target being spectator, not involved in the collision. Express x as a function of Q^2 and $P \cdot q$. Such a scattering is named deep inelastic (DIS) when Q^2 is large (with respect to Λ^2_{QCD}). In the case of DIS at high energy (a few GeV or more) on a proton, x can be interpreted as the momentum fraction carried by a quark (constituent of the proton) scattered by the incoming electron.

Solution

This model is due to Bjorken and Feynman, where the photon scatters off a part of the proton ("parton"), which turns out to be a quark. This quark is almost free during the very short time of the interaction, for $Q^2 \gg \Lambda^2_{QCD}$, the scale which characterizes the scale of hadronization, since at such a large scale, strong coupling is small, due to asymptotic freedom of QCD. This is illustrated by Fig. 5.
From the relation $K - K' + xP = xP'$ we get $(K - K' + xP)^2 = (xP')^2$ which reads

$$q^2 + 2x\, q \cdot P + x^2 M^2 = x^2 M^2$$

and thus

$$\boxed{x = \tfrac{Q^2}{2P\cdot q} \equiv x_{\text{Bjorken}}\,.} \tag{3}$$

This relation allowed to check the model in 1969, since x_{Bjorken} can be measured experimentally (q and thus Q^2 are known since K (beam) and K' (reconstructed out-going e^-) are known, and P is known (target)), so that one has a direct access to x!

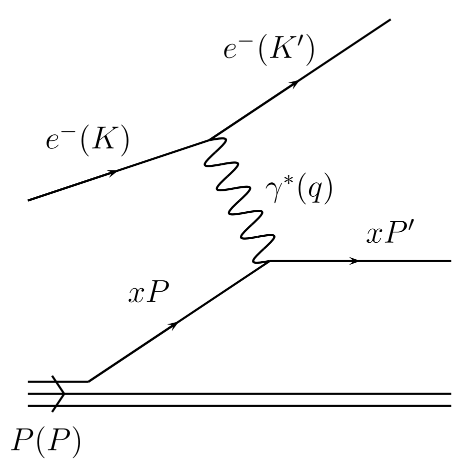

Figure 5: Parton model.

4. Draw, in the Q^2, $P \cdot q$ plane, the lines corresponding to DIS at fixed x, as well as the lines corresponding to quasi-elastic scattering.

Solution

In order to simplify the notations, we now denote $x_{\rm Bj} = x_{\rm Bjorken}$.

Quasi-elastic scattering: corresponds to M' fixed. Since

$$q^2 = -Q^2 = M'^2 - M^2 - 2P \cdot q$$

one has

$$2P \cdot q = Q^2 + M'^2 - M^2$$

which means parallel lines of slope 1/2, the limiting case being $M'^2 = M^2$, corresponding to $x_{\rm Bj} = 1$, i.e. the line passing through 0.

Inelastic scattering: corresponds to $x_{\rm Bj}$ fixed, that is

$$P \cdot q = \frac{Q^2}{2x_{\rm Bj}}$$

which means lines passing through 0, of slope $1/(2x_{\rm Bj})$. This is illustrated in Fig. 6

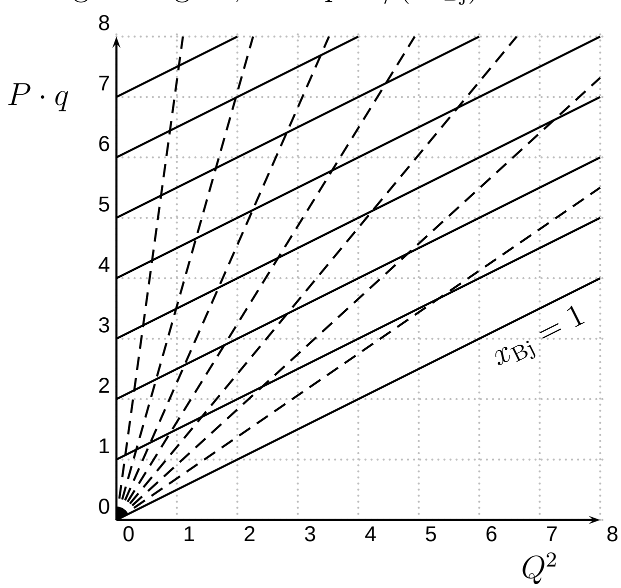

Figure 6: The $(Q^2, P \cdot q)$ plane. Continuous lines: quasi-elastic scattering (M' fixed). Dashed line: inelastic scattering at fixed $x_{\rm Bj}$.

5. The target is assumed to be at rest. Draw, again in the Q^2, $P \cdot q$ plane, the lines corresponding to fixed k' as well as those corresponding to fixed θ. Deduce from that the allowed kinematical region. Compute the maximal value of Q^2 for a given x, in the limit $k \gg M$.

In the fixed target mode, $E = M$ and $p = 0$. From

$$q^2 = -2kk'(1-\cos\theta) \leqslant 0\,,$$

we get

$$k' = \frac{Q^2}{2k(1-\cos\theta)}\,.$$

Since $P \cdot q = M(k - k')$ we obtain

$$P \cdot q = Mk - \frac{M}{2k(1-\cos\theta)}Q^2\,. \tag{4}$$

Fixed k': $P \cdot q$ is therefore fixed, and we get an horizontal line in the $(Q^2, P \cdot p)$ plane. Besides,

$$Q^2 = 2kk'(1-\cos\theta)$$

which thus varies between 0 and $4kk'$.

Fixed θ: we then have a straight line in the $(Q^2, P \cdot p)$ plane, since

$$P \cdot q = Mk - \frac{M}{2k(1-\cos\theta)}Q^2\,.$$

The slope $-\dfrac{M}{2k(1-\cos\theta)}$ varies between $-\infty$ and $-M/(4k)$.
Note that $2P \cdot q = Q^2 + M'^2 + M^2$ so that $P \cdot q \geqslant Q^2/2$, the equality corresponding to $x_{\mathrm{Bj}} = 1$. Combining Eq. (4) with the definition (3) of x_{Bj}, we get

$$Q^2 = 2k^2(1-\cos\theta) - \frac{Q^2}{x_{\mathrm{Bj}}}\frac{k}{M}(1-\cos\theta)$$

and thus

$$Q^2 = \frac{2k^2(1-\cos\theta)}{1 + \frac{k}{Mx_{\mathrm{Bj}}}(1-\cos\theta)}$$

Denoting $C = 1 - \cos\theta$ we have

$$\frac{dQ^2}{dC} = \frac{2k^2}{\left(1 + \frac{k}{Mx_{\mathrm{Bj}}}C\right)^2} > 0$$

so that Q^2 is maximal for C maximal, i.e. $\theta = \pi$. We thus get

$$Q^2_{\mathrm{max}} = \frac{4k^2}{1 + \frac{2k}{Mx_{\mathrm{Bj}}}} \sim 2Mx_{\mathrm{Bj}}\,k$$

in the limit $k \gg M$.
The kinematics is illustrated in Fig. 7.

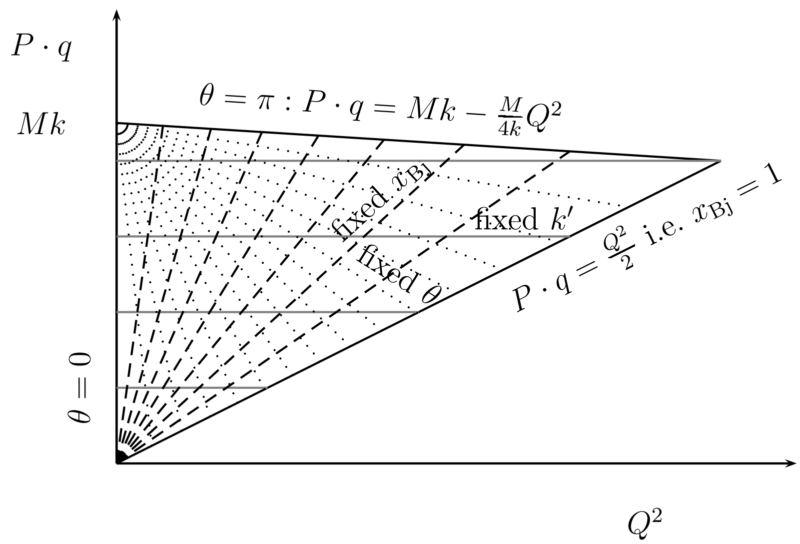

Figure 7: The allowed kinematical range in the $(Q^2, P\cdot q)$ plane, in fixed target mode. Dashed line: inelastic scattering at fixed $x_{\rm Bj}$. Dotted line: fixed θ. Gray lines: fixed k'.

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6. Consider again the previous question, in the case of a collider, in which the target moves at an ultra-relativistic energy $E \gg M$. In the case of HERA (DESY, Hamburg, 1992-2007), $k = 30$ GeV and the target was a proton of energy $E = 800$ GeV. Compute the maximal value of Q^2 for $x = 10^{-4}$. Why is it interesting to accelerate protons? In the future (circa 2030), EIC will scatter electron beams on proton and ions beams.

Solution

In collider mode, neglecting the proton mass, $P = E$, and we get, from the result of question 1.,

$$P \cdot q = 2Ek - (E + E\cos\theta)k' \,.$$

Besides

$$k' = \frac{Q^2}{2k(1-\cos\theta)}$$

so that

$$P \cdot q = 2Ek - \frac{E(1+\cos\theta)}{2k(1-\cos\theta)}Q^2 \,. \tag{5}$$

The ratio $\frac{1+\cos\theta}{1-\cos\theta}$ varies between 0 and $+\infty$ when $\cos\theta$ varies in the range $[-1, 1]$, and so does the slope.
Combining Eq. (5) and the definition (3) one get

$$\left(\frac{1}{2x_{\rm Bj}} + \frac{E}{2k}\frac{1+\cos\theta}{1-\cos\theta}\right)Q^2 = 2Ek$$

and thus

$$Q^2 = \frac{2Ek}{\frac{1}{2x_{\mathrm{Bj}}} + \frac{E}{2k}\frac{1+\cos\theta}{1-\cos\theta}}$$

is maximal when $\frac{1+\cos\theta}{1-\cos\theta}$ is minimal, thus for $\theta = \pi$, i.e.

$$Q^2_{\max} = 4Ekx_{\mathrm{Bj}}\,.$$

The allowed kinematical range is illustrated in Fig. 8.

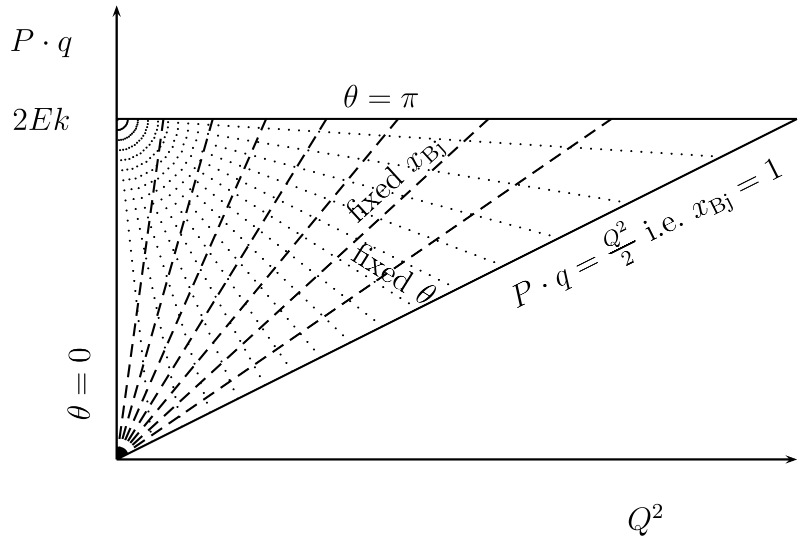

Figure 8: The allowed kinematical range in the $(Q^2, P \cdot q)$ plane, in collider mode. Dashed line: inelastic scattering at fixed x_{Bj}. Dotted line: fixed θ.

At HERA, $Q^2_{\max} = 2 \times 800 \times 10^{-4} \times 30 \simeq 9.6$ GeV^2. For a given x_{Bj}, the maximal value of Q^2 is much higher than in fixed mode, and thus the resolution $\frac{1}{Q}$ is much thiner. Furthermore, Q^2 remains large enough with respect to Λ^2_{QCD} to justify the applicability of perturbative method, even when x_{Bj} is so small that the number of partons (quarks, gluons) is very large, and that collective effects are expected.