M1 General Physics

Particles

Tensors, field strength and Lorentz transform of electromagnetic fields

1 Symmetric and antisymmetric tensors

1.1 Symmetrization, antisymmetrization and covariance

Consider a quadridimensional 2-tensor $M^{\mu\nu}$.

1. Provide a separation of $M^{\mu\nu}$ as a sum of two symmetric (S) and antisymmetric (A) tensors:

$$M^{\mu\nu} = S^{\mu\nu} + A^{\mu\nu}\,. \tag{1}$$

Solution

$$M^{\mu\nu} = \frac{1}{2}\left(M^{\mu\nu} + M^{\nu\mu}\right) + \frac{1}{2}\left(M^{\mu\nu} - M^{\nu\mu}\right) = S^{\mu\nu} + A^{\mu\nu}\,.$$

with

$$S^{\mu\nu} = \frac{1}{2}\left(M^{\mu\nu} + M^{\nu\mu}\right)$$

and

$$A^{\mu\nu} = \frac{1}{2}\left(M^{\mu\nu} - M^{\nu\mu}\right)\,.$$

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2. Recall the way $M^{\mu\nu}$ transform under a arbitrary Lorentz transformation Λ, encoded through its matrix elements $\Lambda^{\rho}{}_{\sigma}$.

Solution

Under a Lorentz transformation Λ, any tensor $M^{\mu\nu}$ transforms according to

$$M'^{\mu\nu} = \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}M^{\mu'\nu'}\,.$$

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3. Show that the decomposition (1) is covariant under Lorentz transformations.

Solution

Under a Lorentz transformation Λ, any tensor $M^{\mu\nu}$ transforms according to

$$M'^{\mu\nu} = \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}M^{\mu'\nu'} \,.$$

On the one hand

$$S'^{\mu\nu} = \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}S^{\mu'\nu'} \,,$$

and thus

$$S'^{\nu\mu} = \Lambda^{\nu}{}_{\mu'}\Lambda^{\mu}{}_{\nu'}S^{\mu'\nu'} = \Lambda^{\nu}{}_{\nu'}\Lambda^{\mu}{}_{\mu'}S^{\nu'\mu'} = \Lambda^{\nu}{}_{\nu'}\Lambda^{\mu}{}_{\mu'}S^{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}S^{\mu'\nu'} = S'^{\mu\nu} \,,$$

where the second equality holds because μ', ν' are dummy summation indexes, and the third one relies on the fact that S is symmetric. The last equality is just a simple reshuffling.
On the other hand

$$A'^{\mu\nu} = \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}A^{\mu'\nu'} \,,$$

and thus

$$A'^{\nu\mu} = \Lambda^{\nu}{}_{\mu'}\Lambda^{\mu}{}_{\nu'}A^{\mu'\nu'} = \Lambda^{\nu}{}_{\nu'}\Lambda^{\mu}{}_{\mu'}A^{\nu'\mu'} = -\Lambda^{\nu}{}_{\nu'}\Lambda^{\mu}{}_{\mu'}A^{\mu'\nu'} = -\Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}A^{\mu'\nu'} = -A'^{\mu\nu} \,,$$

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4. Write the Lorentz transformation of $M^{\mu\nu}$ in a way which makes this separation explicit.

Solution

Using the fact that μ' and ν' are dummy variables, M can be rewritten as

$$\begin{aligned} M'^{\mu\nu} &= \frac{1}{2}\left(\Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'} + \Lambda^{\mu}{}_{\nu'}\Lambda^{\nu}{}_{\mu'}\right)\frac{1}{2}\left(M'^{\mu'\nu'} + M'^{\nu'\mu'}\right) \\ &+ \frac{1}{2}\left(\Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'} - \Lambda^{\nu}{}_{\mu'}\Lambda^{\mu}{}_{\nu'}\right)\frac{1}{2}\left(M'^{\mu'\nu'} - M'^{\nu'\mu'}\right) \end{aligned}$$

which explicitly shows that $S^{\mu'\nu'}$ (first line) transforms through the contraction of the symmetric tensor $\frac{1}{2}\left(\Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'} + \Lambda^{\mu}{}_{\nu'}\Lambda^{\nu}{}_{\mu'}\right)$ while $A^{\mu'\nu'}$ (second line) transforms through the contraction of the antisymmetric tensor $\frac{1}{2}\left(\Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'} - \Lambda^{\nu}{}_{\mu'}\Lambda^{\mu}{}_{\nu'}\right)$.

1.2 Transformation under boosts

We now consider a Lorentz boost of a frame F to a frame F' along the x axis, encoded by $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

5. Recall the explicit expression of Λ.

Solution

$$\left\{\begin{array}{rcrcr} x^{0\prime} &=& \gamma\, x^0 &-& \gamma\beta\, x^1 \\ x^{1\prime} &=& -\gamma\beta\, x^0 &+& \gamma\, x^1 \end{array}\right.$$

and thus

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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6. We focus on the case of a symmetric tensor $S^{\mu\nu}$.

Write the Lorentz transformation of the various components of S under the above boost.

Solution

From $S'^{\mu\nu} = \Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}S^{\mu'\nu'}$ we have, using the fact that S is symmetric,

$$\left\{\begin{array}{rcl} S'^{00} &=& \Lambda^0{}_0\Lambda^0{}_0\, S^{00} + 2\Lambda^0{}_1\Lambda^0{}_0\, S^{01} + \Lambda^0{}_1\Lambda^0{}_1\, S^{11} \\ S'^{01} &=& \Lambda^0{}_0\Lambda^1{}_0\, S^{00} + (\Lambda^0{}_0\Lambda^1{}_1 + \Lambda^0{}_1\Lambda^1{}_0)\, S^{01} + \Lambda^0{}_1\Lambda^1{}_1\, S^{11} \\ S'^{02} &=& \Lambda^0{}_0\, S^{02} + \Lambda^0{}_1 S^{12} \\ S'^{03} &=& \Lambda^0{}_0\, S^{03} + \Lambda^0{}_1 S^{13} \\ S'^{11} &=& \Lambda^1{}_0\Lambda^1{}_0\, S^{00} + 2\Lambda^1{}_0\Lambda^1{}_1\, S^{01} + \Lambda^1{}_1\Lambda^1{}_1\, S^{11} \\ S'^{12} &=& \Lambda^1{}_0\, S^{02} + \Lambda^1{}_1\, S^{12} \\ S'^{13} &=& \Lambda^1{}_0\, S^{03} + \Lambda^1{}_1\, S^{13} \end{array}\right.$$

while S^{22}, S^{23}, S^{33} are invariant. Explicitly, we thus get

$$\left\{\begin{array}{rcl} S'^{00} &=& \gamma^2(S^{00} - 2\beta\, S^{01} + \beta^2\, S^{11}) \\ S'^{01} &=& \gamma^2(-\beta\, S^{00} + (1+\beta^2)S^{01} - \beta\, S^{11}) \\ S'^{02} &=& \gamma(S^{02} - \beta S^{12}) \\ S'^{03} &=& \gamma(S^{03} - \beta S^{13}) \\ S'^{11} &=& \gamma^2(\beta^2 S^{00} - 2\beta\, S^{01} + S^{11}) \\ S'^{12} &=& \gamma(-\beta\, S^{02} + S^{12}) \\ S'^{13} &=& \gamma(-\beta\, S^{03} + S^{13})\,. \end{array}\right.$$

7. We focus on the case of an antisymmetric tensor $A^{\mu\nu}$.

a. What can be said about $A^{00}, A^{11}, A^{22}, A^{33}$ and their transformations?

Solution

$A^{\mu\nu}$ is antisymmetric, therefore $A^{00} = A^{11} = A^{22} = A^{33} = 0$, and this remains valid in any frame obtained through arbitrary Lorentz transformations, as we have shown above.

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b. How does A^{23} transforms?

Solution

Since x^2 and x^3 are invariant under the boost (2), A^{23} is invariant.

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c. Compare the transformation of A^{12}, A^{13} and A^{02}, A^{03} with the transformation of x^1 and x^0. Deduce the transformation of these components.

Solution

The components A^{02} and A^{03} transform as x^0 while A^{12} and A^{13} transform as x^1. Thus,

$$\left\{ \begin{array}{rcrcr} A'^{02} & = & \gamma\, A^{02} & - & \gamma\beta\; A^{12} \\ A'^{12} & = & -\gamma\beta\; A^{02} & + & \gamma\, A^{12} \end{array} \right.$$

and

$$\left\{ \begin{array}{rcrcr} A'^{03} & = & \gamma\, A^{03} & - & \gamma\beta\; A^{13} \\ A'^{13} & = & -\gamma\beta\; A^{03} & + & \gamma\, A^{13} \end{array} \right.$$

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d. Show that A^{01} is invariant under these boosts.

Solution

A^{01} is invariant since

$$A'^{01} = \Lambda^0{}_0\Lambda^1{}_1 A^{01} + \Lambda^0{}_1\Lambda^1{}_0 A^{10} = (\gamma^2 - \gamma^2\beta^2)A^{01} = A^{01},$$

because $\gamma^2(1-\beta^2) = 1\,.$

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2 Lorentz transformations of electromagnetic fields

We now apply the previous results to the field strength

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix},$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields. We use a system of units such that $c = 1$.

8. Show that under the considered boost along x, these fields transform as

$$E'^1 = E^1\,, \tag{2}$$
$$E'^2 = \gamma(E^2 - \beta B^3)\,, \tag{3}$$
$$E'^3 = \gamma(E^3 + \beta B^2) \tag{4}$$

and

$$B'^1 = B^1\,, \tag{5}$$
$$B'^2 = \gamma(B^2 + \beta E^3)\,, \tag{6}$$
$$B'^3 = \gamma(B^3 - \beta E^2)\,. \tag{7}$$

Solution

This is more or less straightforward from the previous questions devoted to the way an antisymmetric tensor gets boosted.

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9. Consider an arbitrary boost along the direction $\vec{n}$ ($\vec{n}^2 = 1$), i.e. with a velocity $\vec{v} = \beta\vec{n}$.

Show that under such a boost,

$$\vec{E}' = (\vec{E}\cdot\vec{n})\vec{n} + \gamma\left[\vec{E} - (\vec{E}\cdot\vec{n})\vec{n}\right] + \gamma\,\vec{v}\wedge\vec{B}\,, \tag{8}$$
$$\vec{B}' = (\vec{B}\cdot\vec{n})\vec{n} + \gamma\left[\vec{B} - (\vec{B}\cdot\vec{n})\vec{n}\right] - \gamma\,\vec{v}\wedge\vec{E}\,. \tag{9}$$

Solution

The easiest way is to write the results (2-7) in an intrinsic form:
First, the components along $\vec{n}$ of both $\vec{E}$ and $\vec{B}$ are invariants, which reads $\vec{E}'\cdot\vec{n})\vec{n} = (\vec{E}\cdot\vec{n})\vec{n}$ and $(\vec{B}'\cdot\vec{n})\vec{n} = (\vec{B}\cdot\vec{n})\vec{n}\,.$
Second, extracting these components through $\vec{E} - (\vec{E}\cdot\vec{n})\vec{n}$ and $\vec{B} - (\vec{B}\cdot\vec{n})\vec{n}$, the remaining components are boosted by a factor γ according to (3,4) and (6,7) respectively.

Third, the $\gamma\beta$ terms in (3,4) and (6,7) respectively read $\gamma\beta\vec{n}\wedge\vec{B}$ and $-\gamma\beta\vec{n}\wedge\vec{E}$ in the case $\vec{n}=\vec{u}_x$. The fact that the boost was along the x direction plays no role in this result, so that we can promote it for arbitrary $\vec{n}$.
Combining this three kinds of contributions leads to the result.

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10. Show that this can be rewritten in the form

$$\vec{E}' = (\vec{E}\cdot\vec{n})\vec{n} + \gamma\left[\vec{n}\wedge(\vec{E}\wedge\vec{n}) + \vec{v}\wedge\vec{B}\right], \tag{10}$$

$$\vec{B}' = (\vec{B}\cdot\vec{n})\vec{n} + \gamma\left[\vec{n}\wedge(\vec{B}\wedge\vec{n}) - \vec{v}\wedge\vec{E}\right]. \tag{11}$$

Hint: use $\vec{a}\wedge(\vec{b}\wedge\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$.

Solution

The double vector product identity reads $\vec{n}\wedge(\vec{E}\wedge\vec{n}) = \vec{E} - (\vec{n}\cdot\vec{E})\vec{n}$, which immediately provides the result.

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11. In the non-relativistic limit, simplify the transformations (8) and (9), keeping linear terms in β.

Solution

The boosted fields read in the non-relativistic limit

$$\vec{E}' = \vec{E} + \vec{v}\wedge\vec{B}, \tag{12}$$

$$\vec{B}' = \vec{B} - \vec{v}\wedge\vec{E}. \tag{13}$$
