DM ECQ

- The system of units is such that $c = 1, \hbar = 1,\ \epsilon_0 = 1,\ \mu_0 = 1$.
- Space coordinates may be freely denoted as (x, y, z) or (x^1, x^2, x^3).
- One may always assume that fields are rapidly decreasing at infinity.

1 Energy-momentum tensor for the electromagnetic field

1. Canonical energy-momentum tensor.

a. We consider a Lagrangian $\mathcal{L}(x) = \mathcal{L}(A_\mu(x), \partial_\nu A_\mu(x))$ constructed from a spin-one field A^μ and its first derivatives, which does not depend explicitly on the space-time position. Based on the Noether's theorem, justify that one can construct a conserved energy-momentum tensor $T^{\mu\nu}$, which reads

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\lambda)} \partial^\nu A_\lambda \, - \, g^{\mu\nu}\mathcal{L}\,. \tag{1}$$

Solution

Using the transformation

$$\begin{aligned} \delta x^\mu(x) &= \text{constant} = \delta x^\mu, \\ \delta\phi &= 0\,, \end{aligned}$$

which obviously leaves the Lagrangian invariant, the Noether's theorem leads to the conservation of the current

$$j^\mu = \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\lambda)} \partial^\nu A_\lambda \, - \, g^{\mu\nu}\mathcal{L} \right) \delta x_\nu \tag{2}$$

for arbitrary δx_ν, thus the conservation of $T^{\mu\nu}$ after factorizing out the arbitrary constant δx_ν. Note the fact that $T^{\mu\nu}$ thus depends on 2 indices. This is exactly the result of the tutorial obtained for a scalar field, except for the sum over the various degrees of freedom A_λ.

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b. In the case of the Lagrangian of a free photon, prove that

$$T^{\mu\nu} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}\,. \tag{3}$$

Solution

We know that the Lagrangian of a pure photon field is

$$\mathcal{L}_{\text{em}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} (\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu)$$

so that

$$\frac{\delta \mathcal{L}_{\rm em}}{\delta(\partial_\mu A_\lambda)} = -F^{\mu\lambda}.$$

Thus,

$$T^{\mu\nu} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}\,.$$

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2. Discuss the symmetry properties of this tensor. Comment.

Solution

The second term in the rhs of Eq. (3) is symmetric. Besides, the first term reads

$$-F^{\mu\lambda}\partial^\nu A_\lambda = -(\partial^\mu A^\lambda - \partial^\lambda A^\mu)\partial^\nu A_\lambda = -(\partial^\mu A^\lambda)(\partial^\nu A_\lambda) + (\partial^\lambda A^\mu)(\partial^\nu A_\lambda)\,.$$

While the first term is obviously symmetric, the second one is not. This is not surprising, as according to the last section of the tutorial "classical fields", a field which carries a spin does not lead to a symmetrical canonical energy-momentum tensor.

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3. Consider the modified energy-momentum tensor

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{4}$$

where $K^{\lambda\mu\nu}$ is antisymmetric in its *first two indices.* This tensor $K^{\lambda\mu\nu}$ is assumed to be built from the field A^μ and its derivatives. Its explicit form plays no role at this stage.

a. Explain why this tensor is an equally good energy-momentum tensor:

(i) Show that $\hat{T}^{\mu\nu}$ is conserved

Solution

One has

$$\partial_\mu \hat{T}^{\mu\nu} = \partial_\mu T^{\mu\nu} + \partial_\mu \partial_\lambda K^{\lambda\mu\nu} = 0$$

where we have used the conservation of the canonical tensor $T^{\mu\nu}$ and the fact that $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices while $\partial_\mu\partial_\lambda$ is symmetric.

(ii) Carefully show that $\hat{T}^{\mu\nu}$ has the same globally conserved energy and momentum as $T^{\mu\nu}$.

Solution

Using $T^{\mu\nu}$ and $\hat{T}^{\mu\nu}$, the conserved charges read respectively

$$P^\nu = \int T^{0\nu} d^3x$$

and

$$\hat{P}^\nu = \int \hat{T}^{0\nu} d^3x = P^\nu + \int \partial_\lambda K^{\lambda 0\nu} d^3x = P^\nu + \int \partial_i K^{i0\nu} d^3x\,.$$

For the last equality, we have used the fact that $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices, so that $K^{00\nu} = 0$. In the last expression, the second term vanishes since it is a total derivative: using the Green-Ostrogradsky theorem, it is equal to the flux of $K^{i0\nu}$ at infinity, which vanishes for sufficiently fast decreasing fields. Thus, $P^\nu = \hat{P}^\nu$.

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b. We now consider the specific case where

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu\,. \tag{5}$$

(i) Show that $\hat{T}^{\mu\nu}$ is now symmetric.

Solution

We now have

$$\hat{T}^{\mu\nu} \;=\; T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} = T^{\mu\nu} + \partial_\lambda (F^{\mu\lambda} A^\nu) = T^{\mu\nu} + F^{\mu\lambda}\partial_\lambda A^\nu$$

where we have used the equation of motion $\partial_\mu F^{\mu\nu} = 0$ (and the fact that $F^{\mu\nu}$ is antisymmetric) in the last equality. Thus, using question 1.b, we have

$$\begin{aligned} \hat{T}^{\mu\nu} &= -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} + F^{\mu\lambda}\partial_\lambda A^\nu \\ &= F^{\mu\lambda} F_\lambda{}^\nu + \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \end{aligned}$$

From this last expression, it is clear that $\hat{T}^{\mu\nu}$ is symmetric.

(ii) Show that one then obtains, for the electromagnetic energy and momentum densities.

Solution

The energy density is given by

$$\mathcal{E} = \hat{T}^{00} = F^{0\lambda} F_\lambda{}^0 + \frac{1}{4} F^{\rho\sigma} F_{\rho\sigma} = F^{0i} F_i{}^0 + \frac{1}{4}\left(2F^{0i} F_{0i} + F^{ij} F_{ij}\right) = \frac{1}{2} F^{0i} F^{0i} + \frac{1}{4} F^{ij} F^{ij}$$

Now, on the one hand, since $F^{i0} = E^i$, we get $F^{0i} F^{0i} = \vec{E}^2$ and on the other hand, since $F^{ij} = -\epsilon_{ijk} B^k$, we have

$$F^{ij} F^{ij} = \epsilon_{ijk} B^k \epsilon_{ij\ell} B^\ell = 2\delta_{k\ell} B^k B^\ell = \vec{B}^2$$

so that

$$\mathcal{E} \;=\; \frac{1}{2}(E^2 + B^2)\,,$$

which is the usual energy density. The momentum density is given by

$$\mathcal{S}^i = \hat{T}^{0i} = F^{0\lambda}F_\lambda{}^i = F^{0k}F_k{}^i = -E^k\epsilon_{ki\ell}B^\ell = (\vec{E}\times\vec{B})^i\,,$$

which is the usual Poynting vector.

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2 Scale invariance

Consider the scalar Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\lambda}{4!}\phi^4. \tag{6}$$

1. Write the Euler-Lagrange equation associated to this Lagrangian. Comment.

Solution

The Euler-Lagrange equation reads

$$\frac{\delta\mathcal{L}(x)}{\delta\phi(x)} - \partial_\mu\frac{\delta\mathcal{L}(x)}{\delta(\partial_\mu\phi(x))} = 0\,.$$

Since

$$\frac{\delta\mathcal{L}(x)}{\delta\phi(x)} = -\frac{\lambda}{3!}\phi^3$$

and

$$\frac{\delta\mathcal{L}(x)}{\delta(\partial_\mu\phi(x))} = \partial^\mu\phi$$

the Euler-Lagrange equation reads

$$\Box\phi + \frac{\lambda}{3!}\phi^3 = 0\,.$$

It reduces as expected to the massless Klein-Gordon equation when $\lambda = 0$.

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2. In analogy with classical mechanics, one defines the momentum Π of the field ϕ as

$$\Pi = \frac{\delta\mathcal{L}}{\delta(\partial_0\phi)}\,. \tag{7}$$

Explain this analogy, and compute Π for the Lagrangian (6).

Solution

In classical mechanics, the momentum is defined as

$$p = \frac{\partial L}{\partial \dot{q}}\,,$$

so that replacing the single degree of freedom q by the infinite degrees of freedom encoded in $\phi(x)$, and the derivative by a functional derivative, the generalization is obvious.
For the Lagrangian (6), one gets

$$\Pi = \partial_0 \phi\,.$$

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3. In this question, we focus on a massive extension of the Lagrangian (6).

a. Mass dimension
(i) What is the mass dimension of $\mathcal{L}$?

Solution

Since the action is dimensionless, and since d^4x has mass dimension -4, the Lagrangian should be of dimension 4.

(ii) What is the mass dimension of the field ϕ?

Solution

Since the kinetic term of Lagrangian has dimension $[\partial/\partial x]^2[\phi]^2 = M^4$, with $[\partial/\partial x] = M$, the field ϕ has mass dimension 1.

(iii) What is the dimension of the coupling constant λ?

Solution

Since $\mathcal{L}$ has mass dimension 4, and since ϕ has mass dimension 1 so that ϕ^4 has mass dimension 4, λ is dimensionless.

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b. What would be the structure of a mass term (quadratic in ϕ), to be added to the Lagrangian (6), up to a multiplicative constant? What would be the equation of motion of this modified Lagrangian? Comment in the case $\lambda = 0$ and fix the value of the constant. Justify finally that this Lagrangian should read

$$\mathcal{L}_m = \frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4, \tag{8}$$

and write its equation of motion.

Solution

The mass term should be quadratic in ϕ and of mass dimension 4, therefore it should look like $c\,m^2\phi^2$ since ϕ is of mass dimension 1. Thus,

$$\mathcal{L}_m = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + c\,m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

The equation of motion is then

$$\Box\phi - 2c\,m^2\phi + \frac{\lambda}{3!}\phi^3 = 0\,.$$

In the case $\lambda = 0$, in order to get the Klein-Gordon equation as the Euler-Lagrange equation of motion for ϕ, one should take $c = -1/2$. Thus,

$$\mathcal{L}_m = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

The equation of motion is now

$$\Box\phi + m^2\phi + \frac{\lambda}{3!}\phi^3 = 0\,.$$

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4. Energy-momentum tensor

a. Massless case:
(i) Compute the energy-momentum tensor $T^{\mu\nu}$ for the Lagrangian (6) and explain why it is conserved. Check by a direct calculation that it is indeed conserved.

Solution

Since the Lagrangian does not depend explicitly on the space-time coordinates, Noether's theorem states that the energy-momentum tensor

$$T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial^\nu\phi \,-\, g^{\mu\nu}\mathcal{L}\,.$$

is conserved. It reads explicitly

$$T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi - g^{\mu\nu}\left(\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - \frac{\lambda}{4!}\phi^4\right).$$

Indeed, a direct computation show that

$$\partial_\mu T^{\mu\nu} = (\Box\phi)(\partial^\nu\phi) + (\partial^\mu\phi)(\partial_\mu\partial^\nu\phi) - (\partial_\nu\partial_\rho\phi)(\partial^\rho\phi) + \frac{\lambda}{3!}\phi^3\partial^\nu\phi \;=\; \left(\Box\phi + \frac{\lambda}{3!}\phi^3\right)\partial^\nu\phi = 0$$

after using the equation of motion derived in question 1.

(ii) Compute the trace $T^\mu{}_\mu$ of the energy-momentum tensor.

Solution

One gets

$$T^\mu{}_\mu = (\partial^\mu\phi)(\partial_\mu\phi) - 2(\partial^\rho\phi)(\partial_\rho\phi) + \frac{\lambda}{3!}\phi^4 = -(\partial^\mu\phi)(\partial_\mu\phi) + \frac{\lambda}{3!}\phi^4\,.$$

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b. Massive case:
(i) Compute the energy-momentum tensor in the case of the massive Lagrangian (8). What is the value of its divergence? Obtain this result by a direct calculation.

Solution

One gets immediately, from question 6. a., modifying appropriately the Lagrangian,

$$T_m^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi - g^{\mu\nu}\left(\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right).$$

Obviously, Noether's theorem states that its divergence is 0. Directly, one has

$$\begin{aligned}\partial_\mu T_m^{\mu\nu} &= (\Box\phi)(\partial^\nu\phi) + (\partial^\mu\phi)(\partial_\mu\partial^\nu\phi) - (\partial_\nu\partial_\rho\phi)(\partial^\rho\phi) + m^2\phi\,\partial^\nu\phi + \frac{\lambda}{3!}\phi^3\partial^\nu\phi \\ &= \left((\Box + m^2)\phi + \frac{\lambda}{3!}\phi^3\right)\partial^\nu\phi = 0\end{aligned}$$

after using the equation of motion derived in question 3.b.

(ii) Compute the trace $(T_m)^\mu{}_\mu$ of the energy-momentum tensor.

Solution

One gets this time

$$(T_m)^\mu{}_\mu = -(\partial^\mu\phi)(\partial_\mu\phi) + 2m^2\phi^2 + \frac{\lambda}{3!}\phi^4\,.$$

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5. We now focus on the scale transformation (or dilation), defined as

$$x \;\to\; x' = bx\,, \tag{9}$$

$$\phi(x) \;\to\; \phi'(x') = \frac{\phi(x)}{b}\,. \tag{10}$$

a. Show that under such a dilation, the Lagrangian (6) is modified as

$$\mathcal{L}_b = \frac{1}{b^4}\mathcal{L}\,. \tag{11}$$

Solution

Under such a transformation, $\partial \to \frac{1}{b}\partial$ so that the transformed Lagrangian becomes

$$\mathcal{L}_b = \frac{1}{b^4}\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\lambda}{b^4}\phi^4 = \frac{1}{b^4}\mathcal{L}\,.$$

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b. Deduce that the action built from the Lagrangian (6) is invariant under the transformation (9,10).

Solution

Since $d^4x \to b^4 d^4x$ under this transformation, it is clear that

$$\int d^4x\,\mathcal{L} \to \int b^4 d^4x\,\mathcal{L}_b = \int d^4x\,\mathcal{L}\,.$$

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c. Discuss the scale invariance of the action built from this modified Lagrangian. Comment on the physical reason of such a behavior.

Solution

This new Lagrangian $\mathcal{L}_m$ does not anymore transform like $\mathcal{L}$ because of the mass term which has a scale, and breaks the scale invariance of the action: this is due to the presence of the dimensional scale provided by the mass.

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6. Under a given transformation acting a priori both on space-time and field, we recall the standard notation

$$\begin{aligned} x'^\mu &= x^\mu + \delta x^\mu, \\ \phi'(x') &= \phi(x) + \delta\phi(x)\,. \end{aligned} \tag{12}$$

a. Show that for an infinitesimal dilation, one has

$$\delta x_\mu = \epsilon x_\mu\,, \tag{13}$$

$$\delta\phi = -\epsilon\phi\,. \tag{14}$$

Solution

With $b = 1+\epsilon$, the scale transformation (9,10) reads, for $|\epsilon| \ll 1$,

$$\begin{aligned} x &\to x' = (1+\epsilon)x\,, \\ \phi(x) &\to \phi'(x') = \frac{\phi(x)}{1+\epsilon} \sim \phi(x) - \epsilon\phi(x)\,, \end{aligned}$$

leading to the expected result for δx_μ and $\delta\phi$.

b. Construct the conserved current built from the above transformation acting on the Lagrangian (6), and deduce that the current

$$J^\mu = -\phi\partial^\mu\phi - \left(\partial^\mu\phi\,\partial^\nu\phi - g^{\mu\nu}\left(\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - \frac{\lambda}{4!}\phi^4\right)\right)x_\nu \tag{15}$$

is conserved. Simplify the expression of this current by using the energy-momentum tensor. Show the conservation of this current directly.

Solution

The action built from the Lagrangian (6) being scale invariant, one should simply use the Noether theorem, which states that the current

$$\begin{aligned} j^\mu &= \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi - \left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial^\nu\phi\;-\;g^{\mu\nu}\mathcal{L}\right)\delta x_\nu \\ &= -\epsilon\phi\,\partial^\mu\phi - \left(\partial^\mu\phi\,\partial^\nu\phi - g^{\mu\nu}\left(\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - \frac{\lambda}{4!}\phi^4\right)\right)\epsilon x_\nu \end{aligned}$$

which implies that, ϵ being arbitrary, the current (15) is conserved. Note that it can be rewritten as

$$J^\mu = -\phi\partial^\mu\phi - T^{\mu\nu}x_\nu$$

Directly, one has

$$\partial_\mu J^\mu \;=\; -(\partial^\mu\phi)(\partial_\mu\phi) - \phi\Box\phi - (\partial_\mu T^{\mu\nu})x_\nu - T^\mu{}_\mu = -(\partial^\mu\phi)(\partial_\mu\phi) - \phi\Box\phi - T^\mu{}_\mu$$

after using the conservation of the energy-momentum tensor. Besides, using the result of question 4.a.(i), one obtains

$$\begin{aligned} \partial_\mu J^\mu &= -(\partial^\mu\phi)(\partial_\mu\phi) - \phi\Box\phi + (\partial^\mu\phi)(\partial_\mu\phi) - 2m^2\phi^2 - \frac{\lambda}{3!}\phi^4 \\ &= -\phi\Box\phi - \frac{\lambda}{3!}\phi^4 = -\phi\left(\Box\phi + \frac{\lambda}{3!}\phi^3\right) = 0\,, \end{aligned}$$

after using the equation of motion.

c. In the case of the massive Lagrangian (8), explain why the generalization of the current (15) is still of the form

$$J^\mu_m = -\phi\partial^\mu\phi - T^{\mu\nu}_m x_\nu\,.$$

Is it still conserved? Compute its divergence.

Solution

We now have

$$\begin{aligned} j^\mu_m &= \frac{\delta \mathcal{L}_m}{\delta(\partial_\mu \phi)}\delta\phi - \left(\frac{\delta \mathcal{L}_m}{\delta(\partial_\mu \phi)}\partial^\nu\phi \;-\; g^{\mu\nu}\mathcal{L}_m\right)\,\delta x_\nu \\ &= -\epsilon\phi\partial^\mu\phi - T^{\mu\nu}_m \epsilon x_\nu \\ &= -\epsilon\phi\,\partial^\mu\phi - \left(\partial^\mu\phi\,\partial^\nu\phi - g^{\mu\nu}\left(\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right)\right)\epsilon x_\nu \end{aligned}$$

and thus

$$\begin{aligned} J^\mu_m &= -\phi\,\partial^\mu\phi - \left(\partial^\mu\phi\,\partial^\nu\phi - g^{\mu\nu}\left(\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right)\right)x_\nu \\ &= -\phi\partial^\mu\phi - T^{\mu\nu}_m x_\nu\,. \end{aligned}$$

Its divergence does not vanish since the scale transformation does not leave the action invariant anymore, see 5.c. Let us compute this divergence directly. The first step is the same as in the massless case:

$$\partial_\mu J^\mu_m \;=\; -(\partial^\mu\phi)(\partial_\mu\phi) - \phi\Box\phi - (\partial_\mu T^{\mu\nu})x_\nu - T^\mu{}_\mu = -(\partial^\mu\phi)(\partial_\mu\phi) - \phi\Box\phi - T^\mu{}_\mu$$

after using the conservation of the energy-momentum tensor. Besides, using the result of question 4.b.(i), one now obtains

$$\begin{aligned} \partial_\mu J^\mu_m &= -(\partial^\mu\phi)(\partial_\mu\phi) - \phi\Box\phi + (\partial^\mu\phi)(\partial_\mu\phi) - 2m^2\phi^2 - \frac{\lambda}{3!}\phi^4 \\ &= -\phi\Box\phi - 2m^2\phi^2 - \frac{\lambda}{3!}\phi^4 = -\phi\left(\Box\phi + \frac{\lambda}{3!}\phi^3\right) - m^2\phi^2 = -m^2\phi^2\,, \end{aligned}$$

after using the equation of motion. As expected, it vanishes when $m = 0$.

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7. We wish to construct a modified energy-momentum tensor and a modified dilation current to get a simple relation between the trace of the energy-momentum tensor and divergence of the current, valid for arbitrary values of m.

a. Show that

$$\phi\,\partial^\sigma\phi = -\frac{1}{6}g_{\rho\tau}\left[g^{\tau\sigma}\partial^\rho - g^{\tau\rho}\partial^\sigma\right]\phi^2. \tag{16}$$

Solution

This is obvious, if one starts from the r.h.s., indeed:

$$-\frac{1}{6}g_{\rho\tau}\left[g^{\tau\sigma}\partial^\rho - g^{\tau\rho}\partial^\sigma\right]\phi^2 = -\frac{1}{6}\left[\partial^\sigma - 4\,\partial^\sigma\right]\phi^2 = \frac{1}{2}\partial^\sigma\phi^2 = \phi\,\partial^\sigma\phi\,.$$

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b. Using the two properties $g_{\rho\tau} = \partial_\rho x_\tau$ and $(\partial X)Y = \partial(XY) - X(\partial Y)$, prove that

$$\phi\, \partial^\sigma \phi = \frac{1}{6} x_\tau \left[g^{\tau\sigma} \Box \phi^2 - g^{\tau\rho} \partial_\rho \partial^\sigma \phi^2 \right] - \partial_\rho X^{\sigma\rho} \tag{17}$$

with

$$X^{\sigma\rho} = \frac{1}{6} \left(x^\sigma \partial^\rho \phi^2 - x^\rho \partial^\sigma \phi^2 \right). \tag{18}$$

Solution

One gets

$$\begin{aligned} -\frac{1}{6} g_{\rho\tau} \left[g^{\tau\sigma} \partial^\rho - g^{\tau\rho} \partial^\sigma \right] \phi^2 &= -\frac{1}{6} (\partial_\rho x_\tau) \left[g^{\tau\sigma} \partial^\rho - g^{\tau\rho} \partial^\sigma \right] \phi^2 \\ &= \frac{1}{6} x_\tau \partial_\rho \left[g^{\tau\sigma} \partial^\rho \phi^2 - g^{\tau\rho} \partial^\sigma \phi^2 \right] - \frac{1}{6} \partial_\rho \left[x^\sigma \partial^\rho \phi^2 - x^\rho \partial^\sigma \phi^2 \right] \\ &= \frac{1}{6} x_\tau \left[g^{\tau\sigma} \Box \phi^2 - g^{\tau\rho} \partial_\rho \partial^\sigma \phi^2 \right] - \partial_\rho X^{\sigma\rho}. \end{aligned}$$

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c. We define the modified dilation current as

$$\tilde{J}^\sigma_m = J^\sigma_m - \partial_\rho X^{\sigma\rho}. \tag{19}$$

Show that

$$\tilde{J}^\sigma_m = -\theta^{\sigma\tau}_m x_\tau \tag{20}$$

with the modified energy-momentum tensor

$$\theta^{\sigma\tau}_m = T^{\sigma\tau}_m + R^{\sigma\tau}, \tag{21}$$

where

$$R^{\sigma\tau} = \frac{1}{6} \left[g^{\sigma\tau} \Box \phi^2 - \partial^\sigma \partial^\tau \phi^2 \right]. \tag{22}$$

Solution

We have, combining 6.c. and 7.b.,

$$\begin{aligned} J^\sigma_m &= -\phi \partial^\sigma \phi - T^{\sigma\tau}_m x_\tau \\ &= -\frac{1}{6} x_\tau \left[g^{\tau\sigma} \Box \phi^2 - g^{\tau\rho} \partial_\rho \partial^\sigma \phi^2 \right] + \partial_\rho X^{\sigma\rho} - T^{\sigma\tau}_m x_\tau \end{aligned}$$

and thus

$$\tilde{J}^\sigma_m = J^\sigma_m - \partial_\rho X^{\sigma\rho} = -\theta^{\sigma\tau}_m x_\tau.$$

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d. Compare the divergence of this modified current with the one of J_m^σ.

Solution

The tensor $X^{\sigma\rho}$ is antisymmetric, therefore, since $\partial_\sigma\partial_\rho$ is symmetric, we have $\partial_\sigma\partial_\rho X^{\sigma\rho} = 0$, so that $\partial_\sigma \tilde{J}_m^\sigma = \partial_\sigma J_m^\sigma$.

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e. Show that the modified energy-momentum tensor $\theta_m^{\sigma\tau}$ has the same key properties as $T_m^{\sigma\tau}$:

(i) Show that $\theta_m^{\sigma\tau}$ is conserved.

Solution

The tensor $R^{\sigma\tau}$ is conserved, since

$$\partial_\sigma R^{\sigma\tau} = \frac{1}{6}\partial_\sigma \left[g^{\sigma\tau}\Box\phi^2 - \partial^\sigma\partial^\tau\phi^2\right] = \partial^\tau\Box\phi^2 - \partial^\tau\Box\phi^2 = 0.$$

This implies that $\theta_m^{\sigma\tau} = T_m^{\sigma\tau} + R^{\sigma\tau}$ is conserved since $T_m^{\sigma\tau}$ is conserved.

(ii) Show that $\theta_m^{\sigma\tau}$ is symmetric.

Solution

The tensor $R^{\sigma\tau}$ is obviously symmetric. Besides, $T_m^{\sigma\tau}$ is symmetric, either from its explicit expression, see 4.b.(i), or from the fact that this is automatic for a spinless field from the conservation of the Noether current built from Lorentz invariance, see tutorial. Therefore, $\theta_m^{\sigma\tau}$ is symmetric.

For the next two questions, since the space volume V extends to infinity, for convenience, it can be taken to have the form of a box with positions of rectangle boundaries at $x_i = \pm\infty$.

(iii) Show that $\theta_m^{\sigma\tau}$ and $T_m^{\sigma\tau}$ lead to the same total 4-momentum.

Solution

The two total 4-momenta, associated with $T_m^{\sigma\tau}$ and $\theta_m^{\sigma\tau}$, respectively read

$$P^\tau = \int_V d^3x\, T_m^{0\tau}$$

and

$$\tilde{P}^\tau = \int_V d^3x\, \theta_m^{0\tau}.$$

Let us consider their difference

$$\tilde{P}^\tau - P^\tau = \int_V d^3x\, R^{0\tau} = \int_V d^3x\, \frac{1}{6}\left[g^{0\tau}\Box\phi^2 - \partial^0\partial^\tau\phi^2\right].$$

- First case: $\tau = 0$, then obviously the integrand vanishes, and thus $\tilde{P}^0 - P^0 = 0\,.$
- Second case: $\tau = i \neq 0$, then, assuming that the space volume is a box, and denoting $d^3x = dx_i d^2S$, and $S(x_i)$ a rectangle in the plane orthogonal to the direction i, at position x_i,

$$\begin{aligned}\int_V d^3x\, R^{0i} &= -\frac{1}{6}\int_V d^3x\, \partial^0\partial^i\phi^2 \\ &= -\frac{1}{6}\partial^0\int_V d^3x\, \partial^i\phi^2 \\ &= -\frac{1}{6}\partial^0\left[\int_{S(x_i=+\infty)} d^2S\,\phi^2 - \int_{S(x_i=-\infty)} d^2S\,\phi^2\right] = 0\end{aligned}$$

using the fact that ϕ is supposed to be fast decreasing at infinity, so that $\tilde{P}^i - P^i = 0\,.$
Thus, $\tilde{P}^\tau = P^\tau$.

(iv) Show that $\theta_m^{\sigma\tau}$ and $T_m^{\sigma\tau}$ lead to the same total angular momentum.

Solution

The two total angular momentum, associated with $T_m^{\sigma\tau}$ and $\theta_m^{\sigma\tau}$, respectively read

$$M^{\mu\nu} = \int_V d^3x\left[x^\mu T_m^{0\nu} - x^\nu T_m^{0\mu}\right]$$

and

$$\tilde{M}^{\mu\nu} = \int_V d^3x\left[x^\mu \theta_m^{0\nu} - x^\nu \theta_m^{0\mu}\right],$$

thus differing from the quantity (which is by the way itself a conserved charge as the difference of two conserved charges)

$$\begin{aligned}\tilde{M}^{\mu\nu} - M^{\mu\nu} &= \int_V d^3x\left[x^\mu R^{0\nu} - x^\nu R^{0\mu}\right] \\ &= \int_V d^3x\left[x^\mu\left(g^{0\nu}\Box\phi^2 - \partial_0\partial^\nu\phi^2\right) - x^\nu\left(g^{0\mu}\Box\phi^2 - \partial_0\partial^\mu\phi^2\right)\right].\end{aligned}$$

- First case: $\mu = i$, $\nu = j \neq i$, then the above integral reads

$$\int_V d^3x\left[-x^i\partial_0\partial^j\phi^2 + x^j\partial_0\partial^i\phi^2\right] = -\partial_0\int_V d^3x\,\partial^j(x^i\phi^2) + \partial_0\int_V d^3x\,\partial^i(x^j\phi^2) = 0\,,$$

indeed each of these two integrals is a total derivative and using the same argument as in the previous question, they both vanish. Thus $\tilde{M}^{ij} - M^{ij} = 0\,.$
- Second case: $\mu = 0$, $\nu = i$, then the above integral reads

$$-\int_V d^3x\, x^0\partial_0\partial^i\phi^2 = -x^0\partial_0\int_V d^3x\,\partial^i\phi^2 = 0$$

again using the same argument (note that since this is a conserved charge, it can thus be computed at an arbitrary time, say $x^0 = 0$, which makes this vanishing straightforward). Thus, $\tilde{M}^{0i} - M^{0i} = 0$.
In conclusion, $\tilde{M}^{\mu\nu} = M^{\mu\nu}$.

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f. Show that the two currents J^σ and $\tilde{J}^\sigma$ have the same associated charge.

Solution

The charge associated to $\tilde{J}^\sigma$ reads

$$\begin{aligned}\int d^3x\,\tilde{J}^0 &= \int_V d^3x\,J^0 - \int_V \partial_\rho X^{0\rho}\,d^3x = \int_V d^3x\,J^0 - \int_V \partial_i X^{0i}\,d^3x \\ &= \int_V d^3x\,J^0 - \int_{S_\infty} X^{0i}\,d^2S_i = \int_V d^3x\,J^0\end{aligned}$$

where we have used the antisymmetry of $X^{\rho\sigma}$ for the second equality, and the Green-Ostrogradsky theorem for the third equality ($d^2\vec{S}$ is the 2d-elementary surface). The final equality uses the fact that field are supposed to be fast decreasing at infinity, so that the flux vanishes on S_∞.

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g. Relation between the trace of $\theta_m^{\sigma\tau}$ and the conservation of $\tilde{J}^\sigma$

(i) Show by a direct computation that the trace of $\theta_m^{\sigma\tau}$ reads

$$(\theta_m)^\mu{}_\mu = m^2\phi^2. \tag{23}$$

Solution

From $\theta_m^{\sigma\tau} = T_m^{\sigma\tau} + R^{\sigma\tau}$ we have

$$(\theta_m)^\mu{}_\mu = (T_m)^\mu{}_\mu + R^\mu{}_\mu$$

with

$$R^\mu{}_\mu = \frac{1}{6}\left[g^\mu{}_\mu \Box\phi^2 - \Box\phi^2\right] = \frac{1}{2}\Box\phi^2 = \frac{1}{2}(\partial_\mu\partial^\mu)\phi^2 = \partial_\mu[\phi\partial^\mu\phi] = (\partial_\mu\phi\,\partial^\mu\phi) + \phi\Box\phi\,.$$

and thus, from 4.b.(ii),

$$\begin{aligned}(\theta_m)^\mu{}_\mu &= -(\partial^\mu\phi)(\partial_\mu\phi) + 2m^2\phi^2 + \frac{\lambda}{3!}\phi^4 + (\partial_\mu\phi\,\partial^\mu\phi) + \phi\Box\phi \\ &= 2m^2\phi^2 + \frac{\lambda}{3!}\phi^4 + \phi\Box\phi = \phi\left[(\Box + m^2)\phi + \frac{\lambda}{3!}\phi^3\right] + m^2\phi^2 = m^2\phi^2\,,\end{aligned}$$

where we have used the equation of motion satisfied by ϕ.

(ii) Relate finally the trace of the modified energy-momentum tensor $\theta_m^{\sigma\tau}$ with the divergence of J_m^σ and check the value of the trace of $\theta_m^{\sigma\tau}$. Conclude by providing a criterion on the tensor $\theta_m^{\sigma\tau}$ for a scalar field theory to be scale invariant.

Solution

From the relation (20) and using the conservation of $\theta_m^{\sigma\tau}$, one immediately gets

$$\partial_\sigma J_m^\sigma = \partial_\sigma \tilde{J}_m^\sigma = -(\theta_m)^\mu{}_\mu\,,$$

and thus, using question 6.c.,

$$(\theta_m)^\mu{}_\mu = m^2\phi^2$$

in agreement with the previous question.
In conclusion,

$$\boxed{\partial_\sigma J_m^\sigma = 0 \Longleftrightarrow (\theta_m)^\mu{}_\mu = 0}$$