# DM ECQ

- The system of units is such that $c = 1, \hbar = 1, \epsilon_0 = 1, \mu_0 = 1$.
- Space coordinates may be freely denoted as $(x, y, z)$ or $(x^1, x^2, x^3)$.
- One may always assume that fields are rapidly decreasing at infinity.

## 1 Energy-momentum tensor for the electromagnetic field

1. Canonical energy-momentum tensor.

a. We consider a Lagrangian $\mathcal{L}(x) = \mathcal{L}(A_\mu(x), \partial_\nu A_\mu(x))$ constructed from a spin-one field $A^\mu$ and its first derivatives, which does not depend explicitly on the space-time position. Based on the Noether's theorem, justify that one can construct a conserved energy-momentum tensor $T^{\mu\nu}$, which reads

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - g^{\mu\nu} \mathcal{L} \,. \tag{1}$$

b. In the case of the Lagrangian of a free photon, prove that

$$T^{\mu\nu} = -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \,. \tag{2}$$

2. Discuss the symmetry properties of this tensor. Comment.

3. Consider the modified energy-momentum tensor

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{3}$$

where $K^{\lambda\mu\nu}$ is antisymmetric in its *first two indices.* This tensor $K^{\lambda\mu\nu}$ is assumed to be built from the field $A^\mu$ and its derivatives. Its explicit form plays no role at this stage.

a. Explain why this tensor is an equally good energy-momentum tensor:

(i) Show that $\hat{T}^{\mu\nu}$ is conserved

(ii) Carefully show that $\hat{T}^{\mu\nu}$ has the same globally conserved energy and momentum as $T^{\mu\nu}$.

b. We now consider the specific case where

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu \,. \tag{4}$$

(i) Show that $\hat{T}^{\mu\nu}$ is now symmetric.

(ii) Show that one then obtains, for the electromagnetic energy and momentum densities.

## 2 Scale invariance

Consider the scalar Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\lambda}{4!}\phi^4. \tag{5}$$

1. Write the Euler-Lagrange equation associated to this Lagrangian. Comment.

2. In analogy with classical mechanics, one defines the momentum $\Pi$ of the field $\phi$ as

$$\Pi = \frac{\delta\mathcal{L}}{\delta(\partial_0\phi)}\,. \tag{6}$$

Explain this analogy, and compute $\Pi$ for the Lagrangian (5).

3. In this question, we focus on a massive extension of the Lagrangian (5).

a. Mass dimension
(i) What is the mass dimension of $\mathcal{L}$?

(ii) What is the mass dimension of the field $\phi$?

(iii) What is the dimension of the coupling constant $\lambda$?

b. What would be the structure of a mass term (quadratic in $\phi$), to be added to the Lagrangian (5), up to a multiplicative constant? What would be the equation of motion of this modified Lagrangian? Comment in the case $\lambda = 0$ and fix the value of the constant. Justify finally that this Lagrangian should read

$$\mathcal{L}_m = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{7}$$

and write its equation of motion.

4. Energy-momentum tensor

a. Massless case:
(i) Compute the energy-momentum tensor $T^{\mu\nu}$ for the Lagrangian (5) and explain why it is conserved. Check by a direct calculation that it is indeed conserved.

(ii) Compute the trace $T^\mu{}_\mu$ of the energy-momentum tensor.

b. Massive case:
(i) Compute the energy-momentum tensor in the case of the massive Lagrangian (7). What is the value of its divergence? Obtain this result by a direct calculation.

(ii) Compute the trace $(T_m)^\mu{}_\mu$ of the energy-momentum tensor.

5. We now focus on the scale transformation (or dilation), defined as

$$
\begin{aligned}
x &\rightarrow x' = bx\,, && (8)\\
\phi(x) &\rightarrow \phi'(x') = \frac{\phi(x)}{b}\,. && (9)
\end{aligned}
$$

a. Show that under such a dilation, the Lagrangian (5) is modified as

$$
\mathcal{L}_b = \frac{1}{b^4}\mathcal{L}\,. \tag{10}
$$

b. Deduce that the action built from the Lagrangian (5) is invariant under the transformation (8,9).

c. Discuss the scale invariance of the action built from this modified Lagrangian. Comment on the physical reason of such a behavior.

6. Under a given transformation acting a priori both on space-time and field, we recall the standard notation

$$
\begin{aligned}
x'^\mu &= x^\mu + \delta x^\mu,\\
\phi'(x') &= \phi(x) + \delta\phi(x)\,.
\end{aligned} \tag{11}
$$

a. Show that for an infinitesimal dilation, one has

$$
\begin{aligned}
\delta x_\mu &= \epsilon x_\mu\,, && (12)\\
\delta\phi &= -\epsilon\phi\,. && (13)
\end{aligned}
$$

b. Construct the conserved current built from the above transformation acting on the Lagrangian (5), and deduce that the current

$$
J^\mu = -\phi\partial^\mu\phi - \left(\partial^\mu\phi\,\partial^\nu\phi - g^{\mu\nu}\left(\frac{1}{2}\partial_\rho\phi\,\partial^\rho\phi - \frac{\lambda}{4!}\phi^4\right)\right)x_\nu \tag{14}
$$

is conserved. Simplify the expression of this current by using the energy-momentum tensor. Show the conservation of this current directly.

c. In the case of the massive Lagrangian (7), explain why the generalization of the current (14) is still of the form

$$
J^\mu_m = -\phi\partial^\mu\phi - T^{\mu\nu}_m x_\nu\,.
$$

Is it still conserved? Compute its divergence.

7. We wish to construct a modified energy-momentum tensor and a modified dilation current to get a simple relation between the trace of the energy-momentum tensor and divergence of the current, valid for arbitrary values of $m$.

a. Show that

$$
\phi\,\partial^\sigma\phi = -\frac{1}{6}g_{\rho\tau}\left[g^{\tau\sigma}\partial^\rho - g^{\tau\rho}\partial^\sigma\right]\phi^2. \tag{15}
$$

b. Using the two properties $g_{\rho\tau} = \partial_\rho x_\tau$ and $(\partial X)Y = \partial(XY) - X(\partial Y)$, prove that

$$\phi\,\partial^\sigma\phi = \frac{1}{6}x_\tau\left[g^{\tau\sigma}\Box\phi^2 - g^{\tau\rho}\partial_\rho\partial^\sigma\phi^2\right] - \partial_\rho X^{\sigma\rho} \tag{16}$$

with

$$X^{\sigma\rho} = \frac{1}{6}\left(x^\sigma\partial^\rho\phi^2 - x^\rho\partial^\sigma\phi^2\right). \tag{17}$$

c. We define the modified dilation current as

$$\tilde{J}^\sigma_m = J^\sigma_m - \partial_\rho X^{\sigma\rho}. \tag{18}$$

Show that

$$\tilde{J}^\sigma_m = -\theta^{\sigma\tau}_m x_\tau \tag{19}$$

with the modified energy-momentum tensor

$$\theta^{\sigma\tau}_m = T^{\sigma\tau}_m + R^{\sigma\tau}, \tag{20}$$

where

$$R^{\sigma\tau} = \frac{1}{6}\left[g^{\sigma\tau}\Box\phi^2 - \partial^\sigma\partial^\tau\phi^2\right]. \tag{21}$$

d. Compare the divergence of this modified current with the one of $J^\sigma_m$.

e. Show that the modified energy-momentum tensor $\theta^{\sigma\tau}_m$ has the same key properties as $T^{\sigma\tau}_m$ :

(i) Show that $\theta^{\sigma\tau}_m$ is conserved.

(ii) Show that $\theta^{\sigma\tau}_m$ is symmetric.

For the next two questions, since the space volume $V$ extends to infinity, for convenience, it can be taken to have the form of a box with positions of rectangle boundaries at $x_i = \pm\infty$.

(iii) Show that $\theta^{\sigma\tau}_m$ and $T^{\sigma\tau}_m$ lead to the same total 4-momentum.

(iv) Show that $\theta^{\sigma\tau}_m$ and $T^{\sigma\tau}_m$ lead to the same total angular momentum.

f. Show that the two currents $J^\sigma$ and $\tilde{J}^\sigma$ have the same associated charge.

g. Relation between the trace of $\theta^{\sigma\tau}_m$ and the conservation of $\tilde{J}^\sigma$

(i) Show by a direct computation that the trace of $\theta^{\sigma\tau}_m$ reads

$$(\theta_m)^\mu{}_\mu = m^2\phi^2. \tag{22}$$

(ii) Relate finally the trace of the modified energy-momentum tensor $\theta^{\sigma\tau}_m$ with the divergence of $J^\sigma_m$ and check the value of the trace of $\theta^{\sigma\tau}_m$. Conclude by providing a criterion on the tensor $\theta^{\sigma\tau}_m$ for a scalar field theory to be scale invariant.